International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 9 Issue 1 ǁ 2021 ǁ PP. 14-16

Consecutive natural numbers whose sum of squares of digits is prime

Charles U. Lonappan

ABSTRACT
This document seeks to find out the relationship between the sum of squares of digits and prime number. The main problem addressed is to find the situations when the sum of squares of two consecutive digits is actually prime. The properties of such numbers are studied in detail. Later this topic is extended to abstract algebra and the properties with field theory are introduced.

Mathematical Subject classification: *11A41*
KEYWORDS:*S(n), Y-number, Y-pair, Z_Y ring.*

Date of Submission: 17-12-2020 Date of acceptance: 02-01-2021

I. INTRODUCTION

Prime numbers have been fascinating throughout history. Euclid's elegant proof that the number of primes is infinite has led to research to find the greater primes than earlier. Here we explore when two consecutive numbers have their sum of squares of digits as prime. The properties shared by such numbers shall be generalized and shall be studied in detail.

II. PRELIMINARIES

Definition 2.1:
If n is a natural number, then S(n) is defined as the sum of squares of digits on n. e.g. S(70)=49, S(100)=1.

Definition 2.2:
If n is a natural number such that S(n) and S(n-1) are both prime, then n is said to be a Y-number. e.g. 12 is a Y-number and is the smallest Y-number.

Definition 2.3:
The number preceding a Y-number is called a Y-1 number and together they shall be called a Y-pair. e.g. 12 is a Y-number, 11 is a Y-1 number and (11,12) is a Y-pair.

Definition 2.4:
The set of d digit number natural numbers is denoted by N_d. e.g. N_2 is the set {10,11...,99}.

Defintion 2.5:
If Z_n is the usual ring of integers under addition modulo n, then the ring when n is a Y-number shall be denoted by Z_Y. Similar definition for Z_{Y-1}.

III. PROPERTIES

Theorem 3.1:

For d≥2, (10^{d-1}+1,10^{d-1}+2) is a Y-pair.

Proof: This trivially follows from the fact that S(10^{d-1}+2) = 5 and S(10^{d-1}+1) = 2 for any d≥2. Both 5 and 2 are prime numbers and hence (10^{d-1}+1,10^{d-1}+2) is a Y-pair.
Q.E.D.

Theorem 3.2:
The number of Y-pairs is infinite and so is the number of Y-numbers.
Proof: This follows from theorem 3.1 since for any large natural number d, $(10^{d-1}+1,10^{d-1}+2)$ is a Y-pair.
Q.E.D.

Theorem 3.3:
If n is a Y-number then ϕ(n) is divisible by 4.
Proof: We prove this by contraposition. Suppose ϕ(n) is not divisible by 4. Then by properties of the Euler's totient function, n is either 1,2 or 4 or of the form p, p^m or $2p^m$ where p is a prime of the form p≡3(mod4) and m is a natural number greater than 1. Obviously 1,2 and 4 cannot be a Y-number. We now consider numbers of the form n= p, p^m or $2p^m$ where p is a prime of the form p≡3(mod4) and m is a natural number greater than 1. When p≡3(mod4), the last digit of p is either 1,3, 7 or 9. The last digit of p^m will be also 1,3,7 or 9. So the last digit of $2p^m$ will be 2,6,4 or 8. So in any case the last digit is not zero. Now we consider the following table:

Case	Number of odd digits of n	Number of even digits of n
1	Odd	Even
2	Odd	Odd
3	Even	Odd
4	Even	Even

In cases 3 and 4, S(n) will be even and S(n-1) will be odd. If S(n) is an even number greater than 2, obviously S(n) is not a prime and hence n cannot be a Y-number. Even if there existed an n with S(n) = 2, then n must begin and end in a 1 (since the only possibility to express 2 as the sum of squares is $2 = 1^2+1^2$ and zero cannot be a last digit). But then S(n-1) will be 1 and hence S(n-1) is not a prime. So n cannot be a Y-number.

In cases 1 and 2, S(n) will be odd and S(n-1) will be even. If S(n-1) is an even number greater than 2, then n cannot be a Y-number. Now suppose that S(n-1)=2. Then there are two possibilities:

(a) n-1 begins and ends in a 1 (not necessarily, but may have zeroes in between the ones e.g. 1001): Then the last digit of n will be 2 and first digit will be 1. Then the sum of digits of n will be 3 and n will be divisible by 3. Then n is the product of at least two different primes. This will lead to a contradiction as then ϕ(n) will be divisible by 4.

(b) n-1 begins with a one and ends in a zero with a '1' and all the rest digits zeroes (e.g. 1010) : Then the last digit of n will be 1 and first digit will be 1. Then the sum of digits of n will be 3 and n will be divisible by 3. Then n is the product of at least two different primes. This will lead to a contradiction as then ϕ(n) will be divisible by 4.

So in all these cases n cannot be a Y-number. So by contraposition, if n is a Y-number then ϕ(n) is divisible by 4.
Q.E.D.

Theorem 3.4:
The last digit of a Y-number cannot be 3,4,...,9.
Proof: Suppose that there exists a natural number n which is a Y-number ending in 3. Then n-1 ends in 2. Let σ denotes the sum of squares of digits excluding the last digit. Then S(n)= σ+9 and S(n-1)= σ+4. Since both S(n) and S(n-1) are greater than 2, it follows that at least one of S(n) and S(n-1) is an even number greater than 2. So n cannot be a Y-number. Similar proof holds for numbers ending in 4,5,..,9.
Q.E.D.

Theorem 3.5:
A Y-number cannot be prime. They are always composite.
Proof: From the previous theorem, a Y-number can end only in 0, 1 or 2. Let us assume that there exists a prime number which is a Y-number. Then it can end only in 0, 1 or 2. Obviously 2 is not a Y-number and if n ends in 0 and 2, the number cannot be a prime. So the only possibility for the last digit of n is 1. Then S(n)-S(n-1) = 1. Then S(n) = 3 and S(n-1) = 2 is the only possibility. The only possibility of S(n)=3 is a number consisting of three '1's and zeros in between (zeros not necessary, e.g. 111 or 1011). Then the sum of the digits of n is 3. So n is divisible by 3, contradicting that n is a prime. We arrived at a contradiction and hence there does not exist a prime Y-number.
Q.E.D.

Theorem 3.6:

Z_Y is not a field.

Proof: We know that Z_n is a field if and only if n is a prime.
Since Y-numbers cannot be a prime, Z_Y cannot be a field.
Q.E.D.

Theorem 3.7:
For any Y-number n, Y-1 is a generator of Z_Y.

Proof: The generators of Z_n are the numbers relatively prime to n.
Any two adjacent natural numbers are relatively prime.
So Y-1 will be a generator for Z_Y.
Q.E.D.

Theorem 3.8:
Defining Z_{Y-1} analogous to Z_Y, Z_{Y-1} X Z_Y is a cyclic group.

Proof: We have the following theorem, "m and n are relatively prime if and only if $Z_m x Z_n$ is isomorphic to Z_{mxn} and consequently cyclic".
As an application of this theorem, Z_{Y-1}X Z_Y is cyclic.
Q.E.D.

Theorem 3.9:
Z_Y[i] is not an integral domain.

Proof: We know that Z_n[i] is an integral domain if and only if Z_n is a field.
But we saw that Z_Y is not a field.
Hence, Z_Y[i] is not an integral domain.
Q.E.D.

REFERENCES

[1]. I.N.Herstein, Abstract Algebra, Wiley, Third Edition, 1996.
[2]. U.M.Swamy and A.V.S.N.Murty, Algebra-Abstract and Modern, Pearson, 2012.
[3]. David Burton, Elementary Number Theory, Mc Graw Hill, Seventh edition, 2017.