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Efficiency Properties of the Stein-Minimax Estimators Under Actual value Prediction

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Abstract
In this paper we formulate Stein–minimax type estimators and compare the performance properties in case of actual value predictions. When, the model is estimated by ordinary least squares, it has been found that least squares predicted is unbiased while minimax and Stein-minimax predictors are not unbiased. The superiority conditions of the estimators have been derived by assuming error distribution to be non-normal.

Keywords: *Linear regression model, minimax estimator, Stein-minimax estimator, Actual value prediction.*

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I. INTRODUCTION

Prediction is an important aspect of relationship analysis in any scientific study. It not only reflects the adequacy of underlying model but also assist in making a suitable choice among the competing models. In the context of linear regression models, the predictions are traditionally obtained either for the average values or for the actual values of the study variable.

Section 3.2, describes the model specification and the predictors. Section 3.3, deals with the properties of the estimators. We whereas in Section 3.4 we have derived the expression for the prediction of actual and mean value of the have derived the expression for the prediction of actual value of the study variable for minimax and Stein-minimax estimator separately and their results are presented in the form of theorems in Section -4. In section-5, the comparative studies have been made. Proofs of the theorems are given in section-6.

II Model Specification and Estimators

Let the true linear regression model is

$$Y = X\beta + \sigma u , \tag{2.1}$$

where Y is a (n ×1) vector of n observations on the variables to be explained, X is a (n×p) matrix of n observations on p explanatory variables, β is a $(p \times 1)$ vector of regression coefficients, σ being unknown scalar and u is a $(n \times 1)$ vector of disturbances with mean zero, variance unity and measures of skewness and kurtosis are γ_1 and (γ_2 +3) respectively.

Application of least squares to (2.1) yield the ordinary least squares estimator as

$$b_0 = \left(X^1 X\right)^{-1} X'Y \quad . \tag{2.2}$$

which is well known to be the best linear unbiased estimator of β , having variance- co-variance matrix.

$$V(b_0) = \sigma^2 \left(X^1 X\right)^{-1}. \tag{2.3}$$

The Stein rule estimator of the regression co-efficient from (2.1) is given by

$$b_s = \left[1 - K\frac{\left(Y - Xb_0\right)'\left(Y - Xb_0\right)}{b_o' X'Xb_0}\right] b_0 . \tag{2.4}$$

where K is any positive non-stochastic scalar characterizing the estimation.

The MILE proposed by Kuks and Olman (1971, 1972) and Kuks (1972) is

$$b^* = D^{-1} X' W^{-1} Y . \tag{2.5}$$

where

$$D = \left(\alpha^{-1}\sigma^{2}T + C\right). \quad (2.6)$$

Using above defined quantities, we obtain minimax estimator

$$b^{*} = \beta + \sigma\left(X^{1}W^{-1}X\right)^{-1}X^{1}W^{-1}u - \sigma^{2}\frac{T}{\alpha}\left(X^{1}W^{-1}X\right)^{-1}\beta - \sigma^{3}\frac{T}{\alpha}X^{1}W^{-1}u\left(X^{1}W^{-1}X\right)^{-2}. \quad (2.7)$$

The Stein-minimax regression coefficient of β is obtained by interchanging b_0 by b^* in (2.4) and is given by

$$b_{s}^{*} = \left[1 - K\frac{\left(Y - Xb^{*}\right)^{1}\left(Y - Xb^{*}\right)}{b^{*}X^{1}Xb^{*}}\right]b^{*}. \quad (2.8)$$

Using above defined quantities, we obtain Stein–minimax estimator

$$\begin{aligned} b_{s}^{*} = {} & \beta + \sigma\left(X^{1}W^{-1}X\right)^{-1}X^{1}W^{-1}u - \sigma^{2}\left[\frac{T}{\alpha}\left(X^{1}W^{-1}X\right)^{-1}\beta + K\frac{u^{1}Bu}{B^{1}X^{1}X\beta}\beta\right] \\ & - \sigma^{3}\left[\frac{T}{\alpha}X^{1}W^{-1}u\left(X'W^{-1}X^{-2}\right) + K\frac{u^{1}Bu}{\beta_{1}X^{1}X\beta}\left(X^{1}W^{-1}X\right)^{-1}X^{1}W^{-1}u\right. \\ & \left. + \frac{2K}{\alpha}\beta^{1}\left[\left(X^{1}X\right)^{-1} - \left(X^{1}W^{-1}X\right)^{-2}\right]\frac{TX^{1}u\beta}{\beta^{1}X^{1}X\beta} - 2K\frac{u^{1}Bu}{\beta^{1}X^{1}X\beta}\beta^{1}X^{1}u.\beta\right] \\ & + \frac{\sigma^{4}}{\beta'X'X\beta}\left[K\frac{U'BU}{\beta'X'X\beta}\beta\left(X'W^{-1}X.X'U - 2\frac{T}{\alpha}\beta'X'X\beta\right) + \frac{2K}{\beta'X'X\beta}.\right. \\ & \left\{\beta'X'u\left(u'Bu\left(X'W^{-1}X\right)^{-1}X'W^{-1}u + \frac{2}{\alpha}\beta'\left(\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}TX'\,u\beta\right)\right)\right\} \\ & + \frac{K}{\alpha}\left(X'W^{-1}X\right)^{-1}\left\{\beta\left(Tu'Bu - \frac{\beta'}{\alpha}\left(X'W^{-1}X\right)^{-1}\left(X'X\right)^{2}\right)\right\} \\ & \left. + 2K\beta'\left\{\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right\}TXU'.X'W^{-1}u\right]. \end{aligned} \quad (2.9)$$

The Predictors

To study the performance properties of these estimators for prediction purposes, let us postulate the following prediction vectors.

$$\hat{T}b^{*} = Xb^{*}. \quad (2.10)$$

$$Tb_{s}^{*} = Xb_{s}^{*}. \quad (2.11)$$

where b^{*}, b_{s}^{*} are defined earlier.

III. Properties of Estimators

In order to study the properties of the estimator, we observe that the exact expressions for the bias vectors mean squared error matrices and risk functions of the least squares estimators from model (2.1) can be easily obtained. However, it is not so with the minimax linear estimator. Therefore, in order to derive the expressions for the bias vectors, mean squared error matrices and risk functions, we introduce the following notations:

$$P_X = X\left(X'X\right)^{-1}X', \quad (3.1)$$

$$M = 1 - P_X. \quad (3.2)$$

Here it is easily seen that

$$M = M' \quad : \quad M'.M = M,$$

$$MX' = X'M = 0$$

Substituting (2.1) in (2.2), we observe that b_o is an unbiased estimator of β with variance – covariance matrix as

$$V(b_o) = E(b_o - \beta)'(b_o - \beta) ,$$
$$= \sigma^2 \overline{P}_X . \quad (3.3)$$

where $\overline{P}_X = \left[\ 1 - X(X'X)^{-1} X'\ \right]$,

The distributional assumption regarding the disturbances does not have any effect on the properties of least squares estimator. Clearly, the minimax estimator and Stein-minimax estimator is a biased estimator.

IV. Predictions

In this section, we have considered prediction of actual value and mean value of the study variable separately.

Actual Value Prediction

To study the impact on the prediction of actual value of the study variable, let us first consider the predictor Y based on Ordinary Least Squares (OLS).

$$\hat{T}_o = Xb_o . \quad (4.1)$$

where b_o is defined in (2.2),

It is easy to see that,

$$E\left[\hat{T}_o - T\right] = E\left[X\beta + \sigma PXu - X\beta - \sigma u\right] = 0 . \quad (4.2)$$

So, the predictors based on OLS from the model are unbiased. But the predictors based on minimax and Stein-minimax estimator from the model are biased. The predictive variable associated with them are given by

$$PV\left(\hat{T}_o\right) = E\left(\hat{T}_o - T\right)'\left(\hat{T}_o - T\right) = \sigma^2 (n-p) \ . \quad (4.3)$$

For the predictors based on minimax and Stein-minimax estimator, we have the following results.

Theorem 4.1: When disturbance are small, the bias vector and predictive risk of predictor $\hat{\mathrm{T}}\mathrm{b}^*$ up to the order of our approximation $\mathrm{o}(\sigma^4)$ are given by

$$\mathrm{PB}\left(\hat{\mathrm{T}}\mathrm{b}^*\right) = -\sigma^2 \frac{\mathrm{T}}{\alpha} \mathrm{X}\left(\mathrm{X}'\mathrm{W}^{-1}\mathrm{X}\right)^{-1} \beta . \quad (4.4)$$

$$\mathrm{PR}\left(\hat{\mathrm{T}}\mathrm{b}^*\right) = \sigma^2 (\mathrm{n}-\mathrm{p}) + \sigma^4 \frac{\mathrm{T}'\mathrm{T}}{\alpha^2} \left(\mathrm{X}'\mathrm{W}^{-1}\mathrm{X}\right)^{-2} \beta'\mathrm{X}'\mathrm{X}\beta . \quad (4.5)$$

Theorem 4.2: When disturbance are small the bias vector and predictive risk of predictor Tb_s^* up to the order of our approximation $0(\sigma^4)$ are given by

$$PB\left(\hat{T}b_s^*\right) = -\sigma^2 \left[\frac{T}{\alpha} X\left(X'W^{-1}X\right)^{-1}\beta + \frac{K}{\beta'X'X\beta} X\beta tr(B)\right]$$
$$- \sigma^3 \left[\frac{K}{\beta'X'X\beta} X\left(X'W^{-1}X\right)^{-1} X'W^{-1}\gamma_1 (I_n * B)e + \sigma^4\left[-2K\gamma_1(I_n * B)e\right]\right]. \quad (4.6)$$

$$PR\left(\hat{T}b_s^*\right) = \sigma^2 (n-p) - \sigma^4 \left[4K\{\gamma_2(I_n * B) + trB + 2B\}\right.$$
$$- \frac{2K}{\beta'X'X\beta}\{\gamma_2(I_n * B) + trB + 2B\} + 2K\left(X'W^{-1}X\right)^{-1} trB \frac{T}{\alpha}$$
$$\left. + \frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta + \frac{K^2}{\beta'X'X\beta}\{\gamma_2 trB(I_n * B) + \mathrm{tr}(B).B + 2\mathrm{tr}B\}\right]. \quad (4.7)$$

From (4.4) and (4.5), it is clearly seen that both the predictors $Tb^{*} and$ Tb_s^{*} are biased. This is in contrast with the unbiasedness of $\hat{T}_o$

V. Comparison

* On comparing the risk associated with ordinary least square estimator $(\hat{T}_0)$ and minimax estimator (T_b^{*}) respectively, we observe that

$$Risk(\hat{T}_0) - Risk(T_b^{*}) = -\sigma^4 \frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-2}\beta'X'X\beta . \qquad (5.1)$$

By observing the above expression, we find that $\hat{T}_0$ is more efficient than T_b^{*} in case of actual value prediction.

* On comparing the risk associated with ordinary least square estimator $(\hat{T}_0)$ and Stein-minimax estimator (Tb_s^{*}) respectively, we observe that

$$\begin{aligned}\text{Risk } (\hat{T}_0) - Risk(\hat{T}b_s^{*}) = \sigma^4 \{4K\{(I_n * B) + trB + 2B\} \\ + \frac{2K}{\beta'X'X\beta}\{\gamma_2(I_n * B) + trB + 2B\} \\ - 2K\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1} trB - \frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta \\ - \frac{K^2}{\beta'X'X\beta}\{\gamma_2 trB(I_n * B) + tr(B).B + 2trB\}\}. \qquad (5.2)\end{aligned}$$

By observing the above expression, we find that Stein-minimax estimator is more efficient than ordinary least square estimator so long as γ_1 and β have same sign otherwise reverse hold true. If either $\gamma_1 = 0 \; or \; \lambda_1 = 1$ then both the estimator have same risk. We find that Stein minimax $(\hat{T}b_s^{*})$ dominate the OLS $(\hat{T}_0)$ with respect to risk criterion if K satisfies.

$$\begin{aligned}0 < K < \sigma^4\Bigg[\Bigg[\frac{2}{\beta'X'X\beta}\left(\gamma_2(I_n * B) + trB + 2B\right) - 4\left(\gamma_2(I_n * B) + trB + 2B\right) \\ - 2\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1} trB\Bigg]\Bigg]\Bigg[\frac{1}{\beta'X'X\beta}\left(\gamma_2 trB(I_n * B) + tr(B).B + 2trB\right)\Bigg]\end{aligned}$$

(5.3)

* On comparing the risk associated with minimax estimator (T_b^{*}) and Stein-minimax (Tb_s^{*}), we observe that

$$\begin{aligned}Risk(\hat{T}_b^{*}) - Risk(\hat{T}b_s^{*}) = \sigma^4 [4K(\gamma_2(I_n * B) + trB + 2B) \\ + \frac{2K}{\beta'X'X\beta}\left(\gamma_2(I_n * B) + trB + 2B\right) - 2K\left(X'W^{-1}X\right)^{-1} trB.\frac{T}{\alpha} \\ - \frac{K^2}{\beta'X'X\beta}\left(\gamma_2 trB(I_n * B) + tr(B).B + 2trB\right)\Bigg]. \qquad (5.4)\end{aligned}$$

Hence, we find that $(Tb_s^{*})$is more efficient than (T_b^{*}) with respect to risk criterion if K satisfies.

$$0 < K < \sigma^4\Bigg[\Bigg[\left(\frac{2}{\beta'X'X\beta}\left(\gamma_2(I_n * B) + trB + 2B\right) + 4\left(\gamma_2(I_n * B) + trB + 2B\right)\right)$$

$$-2\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1}trB\Bigg]\left[\sigma^{4}\left[\frac{1}{\beta'X'X\beta}\left(\gamma_{2}trB\left(I_{n}*B\right)+tr(B).B+2trB\right)\right]\right]^{-1}. \quad (5.5)$$

VI. Proof of the Theorems

Theorem 4.1:

Using (2.7), we get

$$\hat{T}_{b}^{*}=X\beta+\sigma\pi_{1}+\sigma^{2}\pi_{2}+\sigma^{3}\pi_{3}. \quad (6.1)$$

Where,

$$\pi_{1}=X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u, \quad (6.2)$$

$$\pi_{2}=-\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta, \quad (6.3)$$

$$\pi_{3}=-\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-2}X'W^{-1}u, \quad (6.4)$$

Now, subtract (2.1) from (6.1), we get

$$\hat{T}_{b}^{*}-Y=\sigma\pi_{1}^{*}+\sigma^{2}\pi_{2}+\sigma^{3}\pi_{3}. \quad (6.5)$$

where

$$\pi_{1}^{*}=-u\left[1-X\left(X'W^{-1}X\right)^{-1}X'W^{-1}\right],$$

π_2 and π_3 are defined previously as (6.3) and (6.4) respectively. The predictive bias of $\hat{T}_{b}^{*}$ for actual value prediction given as (4.4) in the expression (4.1) can be obtained from

$$PB\left(\hat{T}_{b}^{*}\right)=E\left[\hat{T}_{b}^{*}-Y\right]$$
$$=\sigma E\left[\pi_{1}^{*}\right]+\sigma^{2}E\left[\pi_{2}\right]+\sigma^{3}E\left[\pi_{3}\right]. \quad (6.6)$$

where

$$E\left[\pi_{1}^{*}\right]=-E\left[U\left(1-X\left(X'W^{-1}X\right)^{-1}X'W^{-1}\right)\right]=0.$$

$$E\left[\pi_{2}\right]=-E\left[\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta\right]=-\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta.$$

$$E\left[\pi_{3}\right]=-E\left[\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-2}X'W^{-1}u\right]=0.$$

Next we observe that

$$E\left[\pi_{1}^{*'}\pi_{1}^{*}\right]=E\Big[u'u-u'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u-u'W^{-1}X\left(X'W^{-1}X\right)^{-1}X'u$$
$$+u'W^{-1}X\left(X'W^{-1}X\right)^{-1}X'.X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\Big]$$
$$=\Big[1-X\left(X'W^{-1}X\right)^{-1}X'W^{-1}-W^{-1}X\left(X'W^{-1}X\right)^{-1}X'+W^{-1}X\left(X'W^{-1}X\right)^{-1}X'.$$
$$X\left(X'W^{-1}X\right)^{-1}X'W^{-1}\Big].$$

$$E\left[\pi_{1}^{*'}\pi_{2}\right]=E\left[\frac{T}{\alpha}u'X\left(X'W^{-1}X\right)^{-1}\beta-u'W^{-1}X\left(X'W^{-1}X\right)^{-1}\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta\right]=0.$$

$$E\left[\pi_{2}{}'\pi\right]=E\left[\frac{T'}{\alpha}\beta'\left(X'W^{-1}X\right)^{-1}X'u'-\frac{T'}{\alpha}\beta'\left(X'W^{-1}X\right)^{-1}X'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\right]=0.$$

$$E\left[\pi_1^{*'}\pi_3\right] = E\left[\frac{T}{\alpha}u'X\left(X'W^{-1}X\right)^{-2}X'W^{-1}u - u'W^{-1}X\left(X'W^{-1}X\right)^{-1}X'\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-2}X'W^{-1}u\right],$$

$$= \left[\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-2}X'W^{-1} - \frac{T}{\alpha}W^{-1}X\left(X'W^{-1}X\right)^{-1}X'X\left(X'W^{-1}X\right)^{-2}X'W^{-1}\right]$$

.

$$E\left[\pi_3'\pi_1^*\right] = E\left[\frac{T'}{\alpha}u'W^{-1}X\left(X'W^{-1}X\right)^{-2}X'u - \frac{T'}{\alpha}u''W^{-1}X\left(X'W^{-1}X\right)^{-2}X'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\right]$$

$$\left[\frac{T'}{\alpha}W^{-1}X\left(X'W^{-1}X\right)^{-2}X' - \frac{T'}{\alpha}W^{-1}X\left(X'W^{-1}X\right)^{-2}X'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}\right]$$

.

$$E\left[\pi_2{}'\pi_2\right] = E\left[\frac{T'T}{\alpha^2}\beta'\left(X'W^{-1}X\right)^{-1}X'X\left(X'W^{-1}X\right)^{-1}\beta\right],$$

$$= \frac{T'T}{\alpha^2}\beta'\left(X'W^{-1}X\right)^{-1}X'X\left(X'W^{-1}X\right)^{-1}\beta.$$

Using these expressions in

$$PR\left(\hat{T}_b^*\right) = E\left(\hat{T}_b^* - Y\right)'\left(\hat{T}_b^* - Y\right),$$

$$= \sigma^2 E\left(\pi_1^{*'}\pi_1^*\right) + \sigma^3\left(\pi_1^{*'}\pi_2 + \pi_2{}'\pi_1^*\right) + \sigma^4\left[\pi_1^{*'}\pi_3 + \pi_2{}'\pi_2 + \pi_3{}'\pi_1^*\right]. \qquad (6.7)$$

We get the predictive risk up to order $0\left(\sigma^4\right)$of our approximation as stated (4.5) in Theorem (4.1).

Using (2.9) in the Expression (2.11), we observe that

$$\hat{T}b_s^* = X\beta + \sigma\theta_1 + \sigma^2\theta_2 + \sigma^3\theta_3 + \sigma^4\theta_4. \qquad (6.8)$$

where,

$$\theta_1 = X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u. \qquad (6.9)$$

$$\theta_2 = -\left[\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta + K\frac{u'Bu}{\beta'X'X\beta}X\beta\right]. \qquad (6.10)$$

$$\theta_3 = -\left[\frac{T}{\alpha}X'W^{-1}Xu\left(X'W^{-1}X\right)^{-2} + K\frac{u'Bu}{\beta'X'X\beta}X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u + \frac{2K}{\alpha}\beta'.\right.$$

$$\left. X\left[\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right]\frac{TX'u\beta}{\beta'X'X\beta} - 2K\frac{u'Bu}{\beta'X'X\beta}\beta'X'u.X\beta\right]. \qquad (6.11)$$

$$\theta_4 = \frac{1}{\beta'X'X\beta}\left[K\frac{u'Bu}{\beta'X'X\beta}X\beta\left(X'W^{-1}X.X'u - 2\frac{T}{\alpha}\beta'X'X\beta\right)\right.$$

$$+\frac{2K}{\beta'X'X\beta}\left\{\beta'X'uX\begin{pmatrix} u'Bu\left(X'W^{-1}X\right)^{-1}X'W^{-1}u \\ +\frac{2}{\alpha}X\beta'\left[\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right]TX'u\beta \end{pmatrix}\right\}$$

$$+\frac{K}{\alpha}\left(X'W^{-1}X\right)^{-1}\left\{X\beta\left(Tu'Bu - \frac{\beta'}{\alpha}\left(X'W^{-1}X\right)^{-1}\left(X'X\right)^2\right)\beta\right\}$$

$$\left. + 2K\beta'\left\{\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right\}TX'u'.XX'W^{-1}u\right]. \qquad (6.12)$$

Now subtract (2.1) from (6.8), we get

$$\hat{T}_{b_s}^{*} - Y = \sigma\theta_1^* + \sigma^2\theta_2 + \sigma^3\theta_3 + \sigma^4\theta_4 . \tag{6.13}$$

where $\theta_1^* = -\left[1 - X\left(X'W^{-1}X\right)^{-1} X'W^{-1}\right]u$,

and θ_2, θ_3 and θ_4 are defined previously as (6.10), (6.11) and (6.12) respectively. The predictive bias of $\hat{T}b_s^*$ for actual value prediction given as (4.6) in Theorem 4.2 can be obtained from

$$PB\left(\hat{T}b_s^*\right) = E\left[\hat{T}b_s^* - Y\right],$$

$$= \sigma E\left[\theta_1^*\right] + \sigma^2 E\left[\theta_2\right] + \sigma^3 E\left[\theta_3\right] + \sigma^4 E\left[\theta_4\right]. \tag{6.14}$$

$$E\left[\theta_1^*\right] = -E\left[\left(1 - X\left(X'W^{-1}X\right)^{-1} X'W^{-1}\right)u\right] = 0 . \tag{6.15}$$

$$E\left[\theta_2\right] = -E\left[\frac{T}{\alpha} X\left(X'W^{-1}X\right)^{-1}\beta + K\frac{u'Bu}{\beta'X'X\beta}X\beta\right],$$

$$= -\left[\frac{T}{\alpha} X\left(X'W^{-1}X\right)^{-1}\beta + \frac{K}{\beta'X'X\beta}X\beta\, tr(B)\right]. \tag{6.16}$$

$$E\left[\theta_3\right] = -E\left[\frac{T}{\alpha} X'W^{-1}Xu\left(X'W^{-1}X\right)^{-2} + K\frac{u'Bu}{\beta'X'X\beta}X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\right.$$

$$\left. + \frac{2K}{\alpha}\beta'X\left[\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right]\frac{TX'u\beta}{\beta'X'X\beta} - 2K\frac{u'Bu}{\beta'X'X\beta}\beta'X'u.X\beta\right],$$

$$= -\left[\frac{K}{\beta'X'X\beta}X\left(X'W^{-1}X\right)^{-1}X'W^{-1}\gamma_1\left(I_n * B\right)e - 2K\gamma_2\left(I_n * B\right)e\right]. \tag{6.17}$$

$$E\left(\theta_4\right) = E\left[\frac{1}{\beta'X'X\beta}\left\{K\frac{u'Bu}{\beta X'X\beta}X\beta\left(X'W^{-1}X.X'u - 2\frac{T}{\alpha}\beta'X'X\beta\right)\right.\right.$$

$$+ \frac{2K}{\beta'X'X\beta}\left\{\beta'X'uX\begin{pmatrix} u'Bu\left(XW^{-1}X\right)^{-1}X'W^{-1}u \\ + \frac{2}{\alpha}X\beta'\left(\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right)TX'u\beta \end{pmatrix}\right.$$

$$+ \frac{K}{\alpha}\left(X'W^{-1}X\right)^{-1}\left\{X\beta\left(Tu'Bu - \frac{\beta'}{\alpha}\left(\left(X'W^{-1}X\right)^{-1}\left(X'X\right)^2\right)\beta\right)\right\}$$

$$+ 2K\beta'\left\{\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right\}TX'u'.XX'W^{-1}u\Big\}.$$

$$= \frac{1}{\beta'X'X\beta}\left\{K.\gamma_1\left(I_n * B\right)eX\beta X'W^{-1}X.X' - 2K\frac{T}{\alpha}tr(B)\right.$$

$$+ \frac{2K}{\beta'X'X\beta}\left\{\beta'X'X\left(\gamma_2\left(I_n * B\right) + tr(B) + 2B\right)\left(X'W^{-1}X\right)^{-1}X'W^{-1}\right.$$

$$+ \frac{K}{\alpha}\left(X'W^{-1}X\right)^{-1}\left\{X\beta\left(T.tr(B) - \frac{\beta'}{\alpha}\left(\left(X'W^{-1}X\right)^{-1}\left(X'X\right)^2\right)\beta\right)\right\}$$

$$+ 2k\beta'\left\{\left(X'X\right)^{-1} - \left(XW^{-1}X\right)^{-2}TX'X.X'W^{-1}\right\}. \tag{6.18}$$

Next we observe that,

$$E\left(\theta_1^{*'}\theta_1^*\right) = E\left[u'u - u'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u - u'uW^{-1}X\left(X'W^{-1}X\right)^{-1}X'\right.$$

$$+u'W^{-1}X\left(X'W^{-1}X\right)^{-1}X'.X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\Big].$$
$$\Big[\mathrm{I}-\mathrm{X}\left(\mathrm{X'W^{-1}X}\right)^{-1}\mathrm{X'W^{-1}X}\left(\mathrm{X'W^{-1}X}\right)^{-1}\mathrm{X'}+\mathrm{W^{-1}X}\left(\mathrm{X'W^{-1}X}\right)^{-1}\mathrm{X'.X}\Big]$$
$$\left(\mathrm{X'W^{-1}X}\right)^{-1}\mathrm{X'X}\left(\mathrm{X'W^{-1}X}\right)^{-1}\mathrm{X'W^{-1}}\Big]. \qquad (6.19)$$

$$E\left[\theta_1^*\theta_2\right]=E\left[\frac{T}{\alpha}u'X\left(X'W^{-1}X\right)^{-1}\beta+K\frac{u'Bu}{\beta'X'X\beta}u'X\beta\right.$$
$$-u'W^{-1}X\left(X'W^{-1}X\right)^{-1}X'\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta-u'W^{-1}X\left(X'W^{-1}X\right)^{-1}X'.$$
$$\left.K\frac{u'Bu}{\beta'XX\beta}X\beta\right].$$
$$=\left[\frac{K}{\beta'X'X\beta}\gamma_1\left(I_n*B\right)eX\beta-\frac{K}{\beta'X'X\beta}\gamma_1(I_n*B)eW^{-1}X\left(X'W^{-1}X\right)^{-1}X'X\beta\right]. \qquad (6.20)$$

$$E\left[{\theta_1^*}'\theta_3\right]=E\left[\frac{T}{\alpha}X'W^{-1}Xuu'\left(X'W^{-1}X\right)^{-2}+K\frac{u'Bu}{\beta'X'X\beta}u'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\right.$$
$$\left.+\frac{2K}{\alpha}u'\beta'X\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}\right)\frac{TX'u\beta}{\beta'X'X\beta}-2K\frac{u'Bu}{\beta'X'X\beta}u'\beta'X'uX\beta\right].$$
$$=\left[\frac{T}{\alpha}X'W^{-1}X\left(X'W^{-1}X\right)^{-2}+K.\frac{1}{\beta'X'X\beta}\left[\gamma_2\left(I_n*B\right)tr(B)+2B\right].\right.$$
$$X\left(X'W^{-1}X\right)^{-1}X'W^{-1}+\frac{2\mathrm{K}}{\alpha}\beta'\mathrm{X}\left(\left(\mathrm{X'X}\right)^{-1}-\left(\mathrm{X'W^{-1}X}\right)^{-2}\right)\frac{\mathrm{TX'}\beta}{\beta'\mathrm{X'X}\beta}$$
$$\left.-2K\left(\gamma_2\left(I_n*B\right)+tr(B)+2B\right)\right]. \qquad (6.21)$$

$$E\left[\theta_2'\theta_2\right]=E\left[\frac{T'}{\alpha}\beta'\left(X'W^{-1}X\right)^{-1}X'\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta+K\frac{u'Bu}{\beta'X'X\beta}.\frac{T'}{\alpha}.\right.$$
$$\beta'\left(X'W^{-1}X\right)^{-1}X'X\beta+K\frac{u'Bu}{\beta'X'X\beta}\beta'X'\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta$$
$$\left.+K^2\frac{\left(u'Bu\right)^2}{\left(\beta'X'X\beta\right)^2}\beta'X'X\beta\right],$$
$$=\frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-2}\beta'X'X\beta+K.\frac{1}{\beta'X'X\beta}trB.\frac{T'}{\alpha}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta$$
$$+K.\frac{1}{\beta'X'X\beta}trB.\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta$$
$$+\frac{K^2}{\beta'X'X\beta}\left[\gamma_2tr(B)\left(I_n*B\right)+tr(B).B+2trB\right]$$
$$=\frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-2}\beta'X'X\beta+K\left(\frac{T'}{\alpha}+\frac{T}{\alpha}\right)trB\left(X'W^{-1}X\right)^{-1}$$
$$+\frac{K^2}{\beta'X'X\beta}\left[\gamma_2tr(B)\left(I_n*B\right)+tr(B).B+2trB\right)\Big]. \qquad (6.22)$$

$$E\left[\theta_3'\theta_1^*\right] = E\left[\frac{T'}{\alpha}u'X'W^{-1}X\left(X'W^{-1}X\right)^{-2}u + K\frac{u'Bu}{\beta'X'X\beta}u'uW^{-1}X\left(X'W^{-1}X\right)^{-1}X'\right.$$

$$+\frac{2K}{\alpha}\frac{\beta'u'XT'}{\beta'X'X\beta}u\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}\right)X'\beta - 2K\frac{u'Bu}{\beta'X'X\beta}\beta'X'.uu'XB$$

$$-\frac{T'}{\alpha}u'X'W^{-1}X\left(X'W^{-1}X\right)^{-2}X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u - K\frac{u'Bu}{\beta'X'X\beta}u'W^{-1}X\ .$$

$$\left(X'W^{-1}X\right)^{-1}XX\left(X'W^{-1}X\right)^{-1}XW^{-1}u$$

$$-\frac{2K}{\alpha}\frac{\beta'u'XT}{\beta'X'X\beta}\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}X'\beta X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\right)$$

$$\left.+2K\frac{u'Bu}{\beta'X'X\beta}\beta'X'u'X\beta X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u\right].$$

$$=\frac{T'}{\alpha}X'W^{-1}X\left(X'W^{-1}X\right)^{-2}+K.\frac{1}{B'X'X\beta}\left\{\gamma_2\left(I_n * B\right)+tr(B)+2B\right\}.$$

$$W^{-1}X\left(X'W^{-1}X\right)^{-1}X' + \frac{2K}{\alpha}\frac{\beta'XT}{\beta'X'X\beta}\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}X'B\right)$$

$$-2K.\frac{1}{\beta'X'X\beta}\left(\gamma_2\left(I_n * B\right)+tr(B)+2B\right)\beta'X'X\beta$$

$$-\frac{T'}{\alpha}X'W^{-1}X\left(X'W^{-1}X\right)^{-2}X\left(X'W^{-1}X\right)^{-1}XW^{-1}$$

$$-K.\frac{1}{\beta'X'X\beta}\left(\gamma_2\left(I_n * B\right)+tr(B)+2B\right)W^{-1}X\left(X'W^{-1}X\right)^{-1}X'X.$$

$$\left(X'W^{-1}X\right)^{-1}X'W^{-1}-\frac{2K}{\alpha}\frac{\beta'X'X\beta}{\beta'X'X\beta}+\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}\right)X\left(X'W^{-1}X\right)^{-1}X'W^{-1}$$

$$+2K\frac{1}{\beta'X'X\beta}\left(\gamma_2\left(I_n * B\right)+tr(B)trB\right)B'X'XBX\left(X'W^{-1}X\right)^{-1}X'W^{-1}.$$

$$=\left[\ \frac{T'}{\alpha}X'W^{-1}X\left(X'W^{-1}X\right)^{-2}+\frac{K}{\beta'X'XB}\left(\gamma_2\left(I_n * B\right)+tr(B)+2B\right).\right.$$

$$\left.W^{-1}X\left(X'W^{-1}X\right)^{-1}X'\right]+\frac{2K}{\alpha}T\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}\right)$$

$$-2K\left(\gamma_2\left(I_n * B\right)+tr(B)+2B\right)-\frac{T'}{\alpha}X'W^{-1}X\left(X'W^{-1}X\right)^{-2}X.$$

$$\left(X'W^{-1}X\right)^{-1}X'W^{-1}-K\frac{1}{\beta'X'X\beta}\left(\gamma_2\left(I_n * B\right)+trB+2B\right)W^{-1}X\left(X'W^{-1}X\right)^{-1}.$$

$$X'X\left(X'W^{-1}X\right)^{-1}X'W^{-1}-\frac{2K}{\alpha}T\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}\right)X\left(X'W^{-1}X\right)^{-1}X'W^{-1}$$

$$+2K\left(\gamma_2\left(I_n * B\right)+trB+2B\right)X\left(X'W^{-1}X\right)^{-1}X'W^{-1}]. \qquad (6.23)$$

Using these above expression, we get

$$PR\left(\hat{T}b_s^*\right)=E\left(\hat{T}b_s^*-Y\right)'\left(\hat{T}b_s^*-Y\right).$$

$$=\sigma^2E\left(\theta_1^{*'}\theta_1^*\right)+\sigma^3E\left(\theta_1^{*'}\theta_2+\theta_2^1\theta_1^*\right)+\sigma^4E\left(\theta_1^{*'}\theta_3+\theta_3^1\theta_1^*+\theta_2^{*'}\theta_2\right). \qquad (6.24)$$

We get the prediction risk up to order $0\left(\sigma^4\right)$of our approximation as stated in (4.7) in Theorem (4.2).

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