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A NOTE ON $|\bar{N}, p_n^\alpha; \delta|_k$ SUMMABILITY FACTORS

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***Abstract**:*

In this paper a theorem on $|\bar{N}, p_n^\alpha; \delta|_k$ summability factors, which generalizes a theorem of Hikmet Seyhan Özarslan [2] on $|\bar{N}, p_n^\alpha|_k$ summability factors, is proved.

Keywords: *Infinite series, absolute summability, Minkowski's inequalities.*

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I. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let (p_n) be a sequence with $p_0 > 0, p_n \geq 0$ for $n > 0$ and $P_n = \sum_{v=0}^{n} p_v$. We define

$$p_n^\alpha = \sum_{v=0}^{n} A_{n-v}^{\alpha-1} p_v ; P_n^\alpha = \sum_{v=0}^{n} p_v^\alpha \; (P_{-i}^\alpha = p_{-i}^\alpha = 0, i \geq 1)$$

where

$$A_0^\alpha = 1; \; A_n^\alpha = \frac{(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+n)}{n!}, (\alpha > -1, n = 1,2,3,\ldots).$$

The sequence-to-sequence transformation

$$U_n^\alpha = \frac{1}{P_n^\alpha} \sum_{v=0}^{n} p_v^\alpha s_v$$

defines the sequence (U_n^α) of the $(\bar{N}, p_n^\alpha)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n^α).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n^\alpha|_k, k \geq 1$ if ([1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{(k-1)} |U_n^\alpha - U_{n-1}^\alpha|^k < \infty$$

and it is said to be summable $|\bar{N}, p_n^\alpha; \delta|_k, k \geqq 1$ and $\delta \geq 0$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{(\delta k+k-1)} |U_n^\alpha - U_{n-1}^\alpha|^k < \infty.$$

In the special case when $\delta = 0, \alpha = 0$ (respect. $p_n = 1$ for all values of n) $|\bar{N}, p_n^\alpha; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ (resp. $|C, 1, \delta|_k$) summability.

Seyhan [2] proved the following theorem for $|\bar{N}, p_n^\alpha|_k$ summability factor of infinite series.

Theorem A. Let $k \geq 1$ and $\alpha > -1$. If the sequence (s_n) is bounded and the sequences (λ_n) and (p_n^α) satisfy the following conditions

$$\sum_{n=1}^{\infty} p_n^\alpha |\lambda_n| < \infty \tag{1}$$

$$\sum_{n=1}^{\infty} P_n^\alpha |\Delta\lambda_n| < \infty \tag{2}$$

then the series $\sum a_n \lambda_n P_n^\alpha$ is summable $|\bar{N}, p_n^\alpha|_k$.
The aim of this paper is to generalize Theorem A for $|\bar{N}, p_n^\alpha; \delta|_k$ summability.
Theorem 1. Let $k \geq 1, \delta \geq 0$ and $\alpha > -1$. If the sequence (s_n) is bounded and the sequences (λ_n) and (p_n^α) satisfy the following conditions

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k} p_n^\alpha |\lambda_n| < \infty \tag{3}$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k} P_n^\alpha |\Delta\lambda_n| < \infty \tag{4}$$

and

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha} = O\left\{\left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\delta k} \frac{1}{p_v^\alpha}\right\} \tag{5}$$

then the series $\sum a_n \lambda_n P_n^\alpha$ is summable $|\bar{N}, p_n^\alpha; \delta|_k$ for $k \geq 1$ and $0 \leq \delta \leq \frac{1}{k}$,
Remark. It may be noted that, if we take $\delta = 0$ in this theorem, then we get Theorem A. In this case conditions (3) and (4) reduces to conditions (1) and (2) respectively, and condition (5) reduces to

$$\sum_{n=v+1}^{\infty} \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} = O\left(\frac{1}{P_v^\alpha}\right)$$

which always holds.
We need the following Lemma for the proof of our theorem.
Lemma. If the sequence (λ_n) and (p_n^α) satisfy the conditions (1) and (2) of Theorem A then $P_m^\alpha |\lambda_m| = O(1)$ as $m \to \infty$.

Proof. By Abel's partial summation formula, we have

$$\sum_{n=1}^{m} p_n^\alpha \lambda_n = \sum_{n=1}^{m-1} P_n^\alpha \Delta\lambda_n + P_m^\alpha \lambda_m$$

$$P_m^\alpha |\lambda_m| \leq \sum_{n=1}^{m} p_n^\alpha |\lambda_n| + \sum_{n=1}^{m-1} P_n^\alpha |\Delta\lambda_n| = O(1)$$

Hence $P_m^\alpha |\lambda_m| = O(1)$ as $m \to \infty$.

II. Proof of the Theorem 1

Let (T_n^α) be the sequence of ($\bar{N}, p_n^\alpha$) mean of the series $\sum a_n\lambda_n p_n^\alpha$. Then by definition, we have

$$T_n^\alpha = \frac{1}{P_n^\alpha}\sum_{v=1}^{n} p_v^\alpha \sum_{r=0}^{v} a_r P_r^\alpha \lambda_r$$

$$T_n^\alpha = \frac{1}{P_n^\alpha}\sum_{v=0}^{n} (P_n^\alpha - P_{n-1}^\alpha) a_v P_v^\alpha \lambda_v$$

Then for $n \geq 1$ we have

$$T_n^\alpha - T_{n-1}^\alpha = \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n} P_{v-1}^\alpha P_v^\alpha a_v \lambda_v$$

By Abel's transformation and the formula for difference of products of sequences, we have

$$T_n^\alpha - T_{n-1}^\alpha = \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\left\{\sum_{v=1}^{n-1} \Delta(P_{v-1}^\alpha P_v^\alpha \lambda_v)s_v + P_n^\alpha P_{n-1}^\alpha \lambda_n s_n\right\}$$

$$= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\left\{\sum_{v=1}^{n-1} (P_{v-1}^\alpha P_v^\alpha \lambda_v - P_v^\alpha P_{v+1}^\alpha \lambda_{v+1})s_v\right\} + p_n^\alpha \lambda_n s_n$$

$$= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\left\{\sum_{v=1}^{n-1} (P_v^\alpha - p_v^\alpha)P_v^\alpha \lambda_v s_v - P_v^\alpha(P_v^\alpha + p_{v+1}^\alpha)\lambda_{v+1}s_v\right\} + p_n^\alpha \lambda_n s_n$$

$$= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\left\{\sum_{v=1}^{n-1} P_v^\alpha P_v^\alpha \lambda_v s_v - P_v^\alpha p_v^\alpha \lambda_v s_v - P_v^\alpha P_v^\alpha \lambda_{v+1} s_v - P_v^\alpha p_{v+1}^\alpha \lambda_{v+1} s_v\right\}$$
$$+ p_n^\alpha \lambda_n s_n$$

$$= -\frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha p_v^\alpha \lambda_v s_v + \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha P_v^\alpha \Delta\lambda_v s_v$$
$$- \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha p_{v+1}^\alpha \lambda_{v+1} s_v + p_n^\alpha \lambda_n s_n$$
$$= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha + T_{n,4}^\alpha, \text{ say}$$

by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k + k - 1} |T_{n,r}^\alpha|^k < \infty \text{ for } r = 1,2,3,4.$$

Since (λ_n) is bounded and

$$P_v^\alpha \sum_{n=v+1}^{m+1} \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} = P_v^\alpha \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}^\alpha} - \frac{1}{P_n^\alpha}\right) = P_v^\alpha \left(\frac{1}{P_v^\alpha}\frac{1}{P_{m+1}^\alpha}\right) = O(1), \text{ as } m \to \infty.$$

We have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1} \left|T_{n,1}^\alpha\right|^k$$
$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\left(\frac{1}{P_{n-1}^\alpha}\right)^{k-1} \left\{\sum_{v=1}^{n-1} P_v^\alpha p_v^\alpha |s_v||\lambda_v|\right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\left\{\frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha p_v^\alpha |\lambda_v|\right\}^{k-1} \sum_{v=1}^{n-1} P_v^\alpha p_v^\alpha |\lambda_v|$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha p_v^\alpha |\lambda_v|$$
$$= O(1) \sum_{v=1}^{m} P_v^\alpha p_v^\alpha |\lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}$$
$$= O(1) \sum_{v=1}^{m} p_v^\alpha |\lambda_v| \left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\delta k}$$
$$= O(1), \text{ as } m \to \infty$$

by virtue of condition (3) and (5) of Theorem 1. Again

$$\sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1} \left|T_{n,2}^\alpha\right|^k$$
$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\left(\frac{1}{P_{n-1}^\alpha}\right)^{k-1} \left\{\sum_{v=1}^{n-1} P_v^\alpha P_v^\alpha |s_v||\Delta\lambda_v|\right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\left\{\frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha P_v^\alpha |\Delta\lambda_v|\right\}^{k-1} \sum_{v=1}^{n-1} P_v^\alpha P_v^\alpha |\Delta\lambda_v|$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha P_v^\alpha |\Delta\lambda_v|$$
$$= O(1) \sum_{v=1}^{m} P_v^\alpha P_v^\alpha |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}$$
$$= O(1) \sum_{v=1}^{m} P_v^\alpha |\Delta\lambda_v| \left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\delta k}$$
$$= O(1), \text{ as } m \to \infty$$

by virtue of condition (4) and (5) of Theorem 1. Again similarly, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1} \left|T_{n,3}^\alpha\right|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\left(\frac{1}{P_{n-1}^\alpha}\right)^{k-1} \left\{\sum_{v=1}^{n-1} P_v^\alpha p_{v+1}^\alpha |s_v||\lambda_{v+1}|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\left\{\frac{1}{P_{n-1}^\alpha}\sum_{v=2}^{n} P_{v-1}^\alpha p_v^\alpha |\lambda_v|\right\}^{k-1} \times \sum_{v=2}^{n} P_{v-1}^\alpha p_v^\alpha |\lambda_v|$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}\sum_{v=2}^{n} P_{v-1}^\alpha p_v^\alpha |\lambda_v|$$

$$= O(1) \sum_{v=2}^{m+1} P_{v-1}^\alpha p_v^\alpha |\lambda_v| \sum_{n=v}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha}$$

$$= O(1) \sum_{v=2}^{m} p_v^\alpha |\lambda_v| \left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\delta k}$$

$$= O(1), \text{ as } m \to \infty$$

by virtue of condition (3) and (5) of Theorem 1.
Finally, we have that

$$\sum_{n=1}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1} \left|T_{n,4}^\alpha\right|^k = \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} (P_n^\alpha)^k |s_n|^k |\lambda_n|^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k} (P_n^\alpha)^k \left(\frac{P_n^\alpha}{P_n^\alpha}\right) |\lambda_n|^k$$

$$= O(1) \sum_{n=1}^{m} (P_n^\alpha)^{k-1} p_n^\alpha \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k} |\lambda_n|^k$$

$$= O(1) \sum_{n=1}^{m} (P_n^\alpha |\lambda_n|)^{k-1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k} p_n^\alpha |\lambda_n|$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k} p_n^\alpha |\lambda_n|$$

$$= O(1), \text{ as } m \to \infty$$

by virtue of Lemma and condition (3) of Theorem 1.
Therefore, we get

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1} \left|T_{n,r}^\alpha\right|^k < \infty \text{ for } r = 1,2,3,4.$$

This completes the proof of Theorem 1.

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