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ON (z, x, y) Summability of Jacobi Series

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Abstract-
In this paper a theorem on (z, x, y) summability which generalizes a theorem of BEOHAR [2] is proved.
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I. Introductions: Notations and Definitions

[1.1] Let $\sum a_n$ be a given series with sequence of partial sums $\{\delta_n\}$.
Let $\{x_n\}, \{y_n\}$ be a sequence constant, such that

$$(x^*y)_n = x_0 y_n + x_1 y_{n-1} + \cdots x_n y_0$$
$$= \sum_{m=0}^{n} x_{n-m} y_m$$

Let sequence to sequence transformation, be

$$t_n = \frac{1}{(x * y)_n} \sum_{m=0}^{n} x_{n-m} y_m \delta_m \qquad (1.1)$$

The series $\sum_{m=0}^{\infty} a_m$ is said to be summable (z, x, y) to δ if $t_n \to \delta$ as $n \to \infty$. We shall denote it by

$$\sum_{m=0}^{\infty} a_m = \delta(z, x, y)$$

or

$$\delta_n \to \delta(z, x, y)$$

We shall also use the notations

$$t_n = \frac{1}{D_n} \sum_{m=0}^{n} d_{n-m} y_v \delta_m$$

where

$$d_n = \Delta x_n = x_n - x_{n-m}$$
$$D_n = \sum_{m=0}^{n} \Delta x_m y_{n-m}$$

[1.2] Let $f(\phi) = f(\cos\ \phi), \phi \in [0, \pi]$ be a Lebesgue measurable function such that

$$\int_0^{\pi} f(\phi) R_n^{(\alpha,\beta)}(\cos\ \phi) \sin\ \phi^{2\alpha+1} \cos\ \phi^{2\beta+1} d\phi$$
$$\alpha > -1, \beta > -1$$

exist, where $R_n{}^{(\alpha,\beta)}(\cos\ \phi)$ is the n^{th} Jacobi polynomial of order (α, β).
The Fourier Jacobi series associated with this function is

$$\hat{f}(n) \sim \int_0^{\pi} f(\phi) h_n R_n(\cos\phi) \qquad (1.2.1)$$

where

$$\hat{f}(n) = \int_0^{\pi} f(\phi) R_n(\cos\ \phi) d\mu(\phi)$$

and

$$h_n = \left\{\int_0^{\pi} R_n \cos^2\ d\mu(\phi)\right\}^{-1}$$

$$= \frac{\Gamma(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{(n+\beta+1)\Gamma(n+\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}$$

$$R_n(\cos\ \phi) = x_n^{(\alpha,\beta)}(\cos\ \phi)/x_n^{(\alpha,\beta)}(1)$$

and

$$d\mu(\phi) = (\sin\ \phi/2)^{2\alpha+1}(\cos\ \phi/2)^{2\beta+1} d\phi$$

ASKEY and WAINGER [1] have defined the convolution structure of two function f_1 and f_2 in the following manner.

$$(f_1^* f_2)\phi = \int\ f_1(\phi)(T_\theta f_2(\phi)) d\mu\phi$$

where the generalized translation T_θ is defined by

$$T_\theta f(\phi) = \int\ f(\psi) K(\phi, \theta, \psi) d\mu(\psi)$$

when

$$\int\ R_n(\cos\ \psi) K(\phi, \theta, \psi) = R_n(\cos\ \phi) R_n(\cos\ \psi)$$

and

$$\int\ K(\phi, \theta, \psi) d\mu = 1$$

[1.3] Let

$$\delta_n(\phi) = \sum_{m=0}^{n} \hat{f}(m) h_m P_m(\cos\ \phi)$$

$$= \sum \int_0^{\pi} f(\theta) h_m P_m(\cos\ \phi) P_m(\cos\ \theta) d\mu\theta$$

Now using the orthogonal property, we obtain

$$\delta_n(\phi) - f(\phi) = \sum \{f(\theta) - f(\phi)\}d\mu(\theta) \times$$
$$\times \int_0^{\pi} K(\phi, \theta, \mu)P_n(\cos\ \psi)d\mu(\psi)$$
$$= \int_0^{\pi} L_n \omega_f(\psi) R_n^{(\alpha+1,\beta)}(\cos\psi)d\mu\psi, \tag{1.3.1}$$

where

$$\omega_f(\psi) = T_\psi(f) - f \tag{1.3.2}$$

and

$$\mathrm{L_n} = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \sim n^{\alpha+1}$$

Therefore, we have

$$T_n - f(\phi)$$
$$= \frac{1}{(x^*y)_n} \sum_{k=0}^{n} \{\delta_{n+k}(\phi) - f(\phi)\}dk$$
$$= \frac{1}{(x^*y)_n} \sum_{k=0}^{n} d_k L_{n-k} \int_0^{\pi} \omega(\psi) P_{n-k}^{(\alpha+1,\beta)}(\cos\ \psi)d\psi$$

where

$$\omega(\psi) = \omega_f(\cos\ \psi)(\sin\ \psi/2)^{2\alpha+1}(\frac{\cos\psi}{2)^{2\beta+1}} \tag{1.3.3}$$

INTRODUCTION

[2] In 1985 BEOHAR[2] proved the following theorem.

Theorem A : Let $\{x_n\}$ be a non-negative and non-increasing such that $\left\{x_n n^{-\left(\alpha+\frac{1}{2}\right)}\right\}$ is increasing. if

$$\int_t^{\delta} \frac{|\omega(\psi)|P_{(1/\psi)}\big)d\psi}{\psi^{(\alpha+3)/2}} = O\big(P_{(1/t)^t} t^{(\alpha+1)/2}\big),$$

where

$$\psi = [1/t]$$

and

$$\int_0^{1/\pi} \left|\omega_f(\pi-\psi)\right| \psi^{\beta-\frac{1}{2}} d\psi = O(1) \tag{1.4.1}$$

then for $\alpha > -\frac{1}{2}, \beta > -\frac{1}{2}$, the series

$$t_n = \frac{1}{R_n} \sum_{m=0}^{n} t_m \delta_{n-m}$$

is summable (z, x_n) to $f(\phi)$

Nörlund summability of the series $f(\phi) \sim \sum \hat{f}(n) h_n P_n(\cos\ \phi)$ at the end point has been studied by GUPTA [3] PANDEY AND BEOHAR [4] BEOHAR AND MISHRA [5] and ASKEY AND WAINGER [6] have applied the convolution structure formula and studied the series for the entire rang Q i.e. $[0, \pi]$.

In present paper we study the above theorem for (z, x, y) summability of the series (1.4.1) by applying the convolution structure formula.

We prove the following theorem

[3.1] THEOREM

Let $\{x_n\}$ and $\{y_n\}$ be non-negative and non-increasing such that $\left\{(x^* y)_n n^{-(\alpha+1/2)}\right\}$ is increasing.
If

$$\int_t^s \frac{|\omega(\psi)|(x^* y)\left(\frac{1}{\psi}\right) d\psi}{\psi^{\alpha+3/2}} = O\left((x^* y)\left(\frac{1}{t}\right)^{t^{\alpha+\frac{1}{2}}}\right. \quad (1.5.1)$$

$$\psi = \left[\frac{1}{t}\right]$$

and

$$\int_0^{\frac{1}{n}} \left|\omega_f(\pi - \psi)\right| \psi^{\beta-1/2} d\psi = 0(1) \quad (1.5.2)$$

then, for $\alpha > -\frac{1}{2}, \beta > -\frac{1}{2}$ the series $(1.2 \cdot 1)$ is summable (z, x, y) to $f(\phi)$.

[3.2] For the proof of the theorem, we require the following Lemmas.

LEMMA 1 : Let α, β be real and C a constant, then

$$Z_n(\psi) = \frac{1}{(x^* y)_n} \sum D_k L_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\ \psi)$$

$$Z_n(\psi) = O(n^{2\alpha+2}) \text{ for } 0 < \psi < \frac{1}{n}$$

$$Z_n(\psi) = O\left(\frac{n^{\frac{\alpha+1}{2}}}{(x^* y)_n}\right)\left[\frac{(x \cdot y)^{\left[\frac{\beta}{t}\right]}}{\left(\sin\frac{\psi}{2}\right)^{\alpha+\frac{3}{2}}\left(\cos\frac{\psi}{2}\right)^{\beta+\frac{1}{2}}}\right] + \quad (1.6.1)$$

$$+O\left(n^{\frac{\alpha+1}{2}}\right)\left[\left(\sin\frac{\psi}{2}\right)^{\frac{-\alpha-5}{2}}\left(\cos\frac{\psi}{2}\right)^{\frac{-\beta-\gamma}{2}}\right]$$

for $\frac{1}{n} < \psi < \pi^{\frac{1}{n}}$.

PROOF OF THE LEMMA 1:

We know that, $0 \leqslant \psi \leqslant \frac{1}{n}$

$$x_n^{\alpha+1,\theta}(\cos\ \psi) = O(n^{\alpha+1})$$

Therefore,

$$\begin{aligned} Z_n(\Psi) &= 0\left(\frac{1}{(x^*y)_n}\right)\sum_{k=0}^{n}\ (n-k)^{\alpha+1}(n-k)^{\alpha+1}D_k \\ &= 0\left(\frac{n^{2\alpha+2}}{(x^*y)_n}\sum_{k=0}^{n}\ D_k\right) \\ &= 0(n^{2\alpha+2}) \end{aligned}$$

LEMMA 2: The condition (1.5.1), implies that

$$\int_0^t |\omega(\psi)|d\psi = O(t^{2\alpha+2}) \qquad (1.6.2)$$

PROOF OF THE LEMMA 2:

Implies that condition (1.5.1)

$$\int_0^t |\omega(\psi)|(x^*y)_{\left[\frac{1}{\psi}\right]} = 0\left((x^*y)(t)_{1/t}t^{2\alpha+2}\right)$$

But the integral on the left hand side

$$\geqslant (x^*y)_{\left[\frac{1}{t}\right]}\int_0^t |\omega(\psi)|d\psi$$

Therefore, (1.6.3) is proved

[3.3] PROOF OF THE THEOREM:
we have

$$\begin{aligned} t_n - f(\phi) &= \int_0^{\pi} Z_n(\psi)\omega(\psi)d\psi \\ &= \left\{\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right\} Z_n(\psi)\omega(\psi)d\psi \\ \text{say } &= A_1 + A_2 + A_3 \end{aligned}$$

where

$$A_1 = \int_0^{\frac{1}{n}} O(n^{2\alpha+2})|\omega(\psi)|d\psi$$

by (1.6.1)

$$= O(n^{2a+2})O(\psi^{2a+2})_0^{\frac{1}{n}}$$
$$= O(1) \qquad (1.7.1)$$

Next, we consider

$$A_2 = \int_{\frac{1}{n}}^{\pi-\frac{1}{n}} 0\left(\frac{n^{\alpha+\frac{1}{2}}}{(x^*y)_n}\right)|(\omega(\psi))|(x^*y)_{\left[\frac{1}{\psi}\right]}\Bigg)/$$
$$/\left(\sin\frac{\psi}{2}\right)^{\alpha+3/2}\left(\cos\frac{\psi}{2}\right)^{\beta+\frac{1}{2}} \cdot d\psi +$$
$$+O\left(n^{\alpha-\frac{1}{2}}\int_{\frac{1}{n}}^{\pi-\frac{1}{n}}\left|\ \omega(\psi)\right\}\left|\left(\sin\frac{\psi}{2}\right)^{\alpha-\frac{5}{2}}\left(\cos\frac{\psi}{2}\right)^{-\beta-\frac{3}{2}}\right)d\psi\right.$$
$$= O\left(\frac{n^{\alpha+\frac{1}{2}}}{(x^*y)_n}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}}\frac{1\omega'(\psi)(x^*y)}{\psi^{\alpha+\frac{3}{2}}}\left(\frac{1}{\psi}\right)d\psi +$$
$$+O\left(n^{\alpha+\frac{1}{2}}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}}\frac{1\omega'(\psi)}{\psi^{\alpha+\frac{3}{2}}}d\psi$$
$$= O(1) \text{ by } (1.5.1) \qquad (1.7.2)$$

Also, we get

$$A_3 = O(1) \text{ by } (1.5.2) \qquad (1.7.3)$$

Conclusion: Remark: For $y_n = 1$, our theerem reduces to BEOHAR [2]. Thus the theorem is proved.

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