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On The Lyapunov Theory: Application to a Nonlinear Damped Pendulum.

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Abstract

This brief paper is a review of the classical Lyapunov theory (Lyapunov's second method). The Lyapunov theory uses the concept of a Lyapunov function of the system to draw conclusions on system stability, without actually solving the system differential equation of motion. Some of the primary definitions about this well-established theory are recalled here. The concept is applied to a simple damped nonlinear pendulum . Time evolution of the states (angular position and velocity) are plotted and phase portraits are sketched as well.

Keywords: *Asymptotic Stability, Lyapunov Stability, Lyapunov function, Equilibrium point.*

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I. INTRODUCTION

In Control Systems, stability is the basic requirement before any attempt to make use of a system. One method that can be used for any type of system is the Lyapunov theory [1],[2]. Alexandr Mikhailovich Lyapunov introduced his theory more than a century ago, exactly in 1892. A Lyapunov function whose existence ensures the stability of a system is key to his analysis. That function is an energy-like function (always positive and decreasing with time) along the system's state trajectories. A damped pendulum (a nonlinear system) is chosen as an example to illustrate the concept. Graphical methods (phase portraits) are used to depict the whole picture. It is shown that, for the autonomous system, trajectories sink to the origin for any choice of initial conditions.

1.1. Motivation

The Lyapunov theory has been around for more than a century. However, despite the abundant literature on the issue, many students, especially in French speaking countries, are not so familiar to it. Often, BIBO (Bounded Input, Bounded Output) stability is the main practical stability concept they are used to. It is our strong conviction that, examples of applications such as the pendulum, are likely to spark interest for the theory of Lyapunov. Once the student is acquainted with the notion of Lyapunov stability, he realizes it is not different from asymptotic stability.

1.2. Preliminaries

1.2.1. Abbreviations and Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n dimensional euclidian space
$x(t)$	State vector
$\|x\|$	Euclidian norm
$O_\delta(x_0)$	N neighborhood of x_0 radius $\delta \in \mathbb{R}$
V	Scalar Function

$V: \mathbb{R}^n \times \mathbb{R}\_\rightarrow\_\mathbb{R}$

∇V Gradient of V

BIBO Bounded Input Bounded Output

II. STABILITY HISTORICAL CONCEPTS AND DEFINITIONS

Stability is a fundamental concept in dynamical systems, control theory and its applications. It is believed that, the rigorous mathematical theory of stability had appeared in the course of studying mechanical motions . Some early definitions of stability were given by Joseph L. Lagrange (1736-1802). He stated that a 'stable' position for a pendulum is when its potential energy attains a minimum.
However, the most widely known theory of stability applicable to engineering and many other applied problems is due to Alexander Michailovich Lyapunov (1857-1918) stability theory.

2.1. Basic Definitions

Consider the autonomous system in equation (1).

$$\dot{x} = f(x) \quad \dots \quad (1)$$

Definition 1: Equilibrium Point

If $x_0 = x\,(t_0)$; $f(x_0) = 0$, $t_0 \in \mathbb{R}$, then x_0 is an equilibrium point.

$O_d(x_0) \equiv \{x: \|x - x_0\| < d\}$;$\|x\| = \sqrt{x^T x}$ is the Euclidian norm of $x(t)$.

Definition 2: Stable equilibrium point

If x_0 is an equilibrium point , then $x(t) = x_0$ is a trajectory of the system. An equilibrium point x_0 is **stable** in the sense Lyapunov if :

$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 : x(t_0) \in O_\delta(x_0) \Rightarrow\ x(t) \in O_\varepsilon(x_0)\ , \forall\, t > t_0$.

Any trajectory starting close to the equilibrium point remains close to it.

Definition 3: Asymptotic Stability / Asymptotic Stability in the large

An equilibrium point $x_0 = x(t_0)$ is asymptotically stable in region D if :

$x(t_0) \in D \Rightarrow x(t) \to x_0$ as $t \to +\infty$.

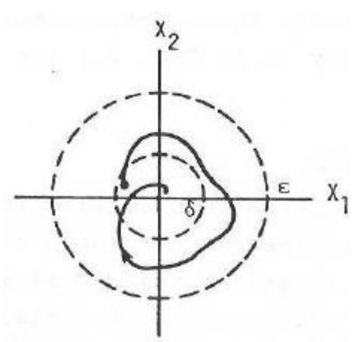

Figure 1.(a) Asymptotic Stability

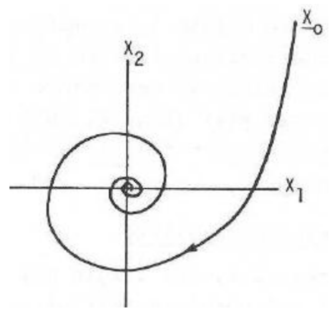

Figure 1.(b) Asymptotic Stability in the large

Definition 4: Positive Definite Function

A scalar function $V = V(x)$ f is said to be **positive definite** if :

i) $V = 0 \Leftrightarrow x = 0;$
ii) $V \neq 0$ if $x \neq 0$
$iii)$ $V > 0$ if $x \neq 0$

Definition 5 : Lyapunov Function

Any scalar positive definite function $V = V(x, t)$ that satisfies condition stated in (1), is called a **Lyapunov function** for the system.

Definition 6 : Lie Derivative

Given a scalar function $V = V(x, t)$, the time derivative can be computed as below.

$$\dot{V}(x) = \frac{dV}{dt} = \nabla V . f(x) \quad \dots \quad (2)$$

The vector $\nabla V\ (x)$ is the gradient of the scalar function $V = V(x)$ and the dot product $\nabla V . f(x)$ of the gradient vector of $V = V(x)$ with the vector field $f(x)$ in equation , is referred to as directional derivative of $V = V(x)$ along the vector field $f(x)$ or the **Lie derivative.**

III. LYAPUNOV'S SECOND METHOD (DIRECT METHOD)

The Lyapunov's Second Method, which is now referred to as the Lyapunov Stability Theorem, makes use of a Lyapunov function to check the stability of an equilibrium point of a system. The system (1) or any system is stable in the sense of Lyapunov if there exists a Lyapunov function for it. This amounts to the statement of the Lyapunov theorem.

3.1. Application to a Pendulum

In this section, we apply the Lyapunov second method to a simple damped pendulum, which is a nonlinear second order system. The states of this system are made up by the angular displacement and its time derivative. Both states are plotted for some given initial conditions.
Consider the pendulum in figure Figure 2. The rope (or rod) of length L is considered to be massless.

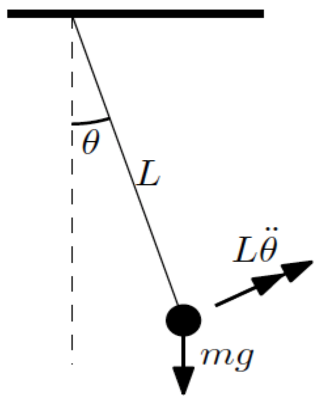

Figure 2 : Simple Pendulum

L is the length of the pendulum

θ (t) the amplitude ; angular position

$\theta(t = 0) = \theta_0 = \frac{\pi}{2},$

$\dot{\theta} =\acute{} \frac{d\theta}{dt}$ position time derivative

$\ddot{\theta} =$ position acceleration

m is the mass

g is gravitational attraction constant

3.2. Motion and Equilibrium Points

There are many ways to derive the equation of motion [3],[4],[5],[6],[7],[8],[9]. The most obvious way is to bring about the issue by the Newton 's approach. Here, one has two options : rotation about an axis normal to the plan of motion , at the point where it is tied; or translational movement (projection on the tangent vector). One can also consider the Euler-Lagrange equation, etc… The autonomous system, is only subject to initial conditions and friction. No external control input is applied.

When using the Euler Lagrange equation (Lagrangian mechanic) , one must write equation (3).

$$\frac{d}{dt}\left(\frac{\partial(K-P)}{\partial\dot{\theta}}\right)-\frac{\partial(K-P)}{\partial\theta}+\frac{d}{dt}\left(\frac{1}{2}k\dot{\theta}^2\right)=0 \qquad \ldots \qquad (3)$$

Where K and P stand for kinetic and potential energy, respectively . The difference K-P is the Lagrangian.

We then get the nonlinear ordinary differential equation (4):

$$mL^2\ddot{\theta}+mgL\sin(\theta)+k\dot{\theta}=0 \qquad \ldots \qquad (4)$$

$(k>0)$

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-\frac{g}{L}sinx_1-\frac{k}{m}x_2\end{cases} \qquad \ldots \qquad (5)$$

$x_1=\theta; x_2=\dot{\theta}\ \ \dot{x}_2=\ddot{\theta}$

To find the equilibrium points, we set $\dot{x}=\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=0;\ \ \dot{x}_1=0$ et $\dot{x}_2=0$.

$x_2=0$ and $-\frac{g}{L}sinx_1-\frac{k}{m}x_2=0$

That is :

$$x_{1\text{eq}}=n\pi \ \ ;\ \ \text{pour } n=0\,;\ \mp1\,;\ \mp2 \qquad \ldots \qquad (6)$$

$x_{2\text{eq}}=0$

$\begin{pmatrix}x_{1\text{eq}}\\ x_{2\text{eq}}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}=x(0)=\ x_0$ is the only equilibrium point of interest here. This is the position of minimum potential energy, which is zero!

3.3. Lyapunov Function for the Pendulum

Consider the Energy as a Lyapunov function. $V(x)=E(x_1,x_2)$=Kinetic+Potential Energy in (7).

$$V(x)=\frac{1}{2}mk^2x_2^2L^2+mgL(1-cosx_1) \qquad \ldots \qquad (7)$$

$V(0)=0$

$$.\dot{V}(x,t)=\frac{dV}{dt}=\dot{x}^T\ \frac{\partial V}{\partial x}=\dot{x}^T\ \nabla V \qquad \ldots \qquad (8)$$

$\nabla V=\begin{pmatrix}mgLsinx_1\\ mk^2L^2x_2\end{pmatrix};$

$\dot{V}(x,t)=\begin{bmatrix}x_2 & -\frac{g}{L}sinx_1-\frac{k}{m}x_2\end{bmatrix}\ \begin{pmatrix}mgLsinx_1\\ mk^2L^2x_2\end{pmatrix}$

$$\dot{V}(x,t)\ =\ -kx_2^2L^2<0 \qquad \ldots \qquad (9)$$

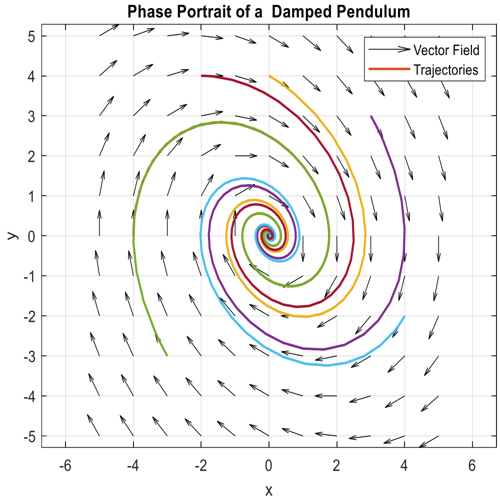

Figure 3. Phase portraits

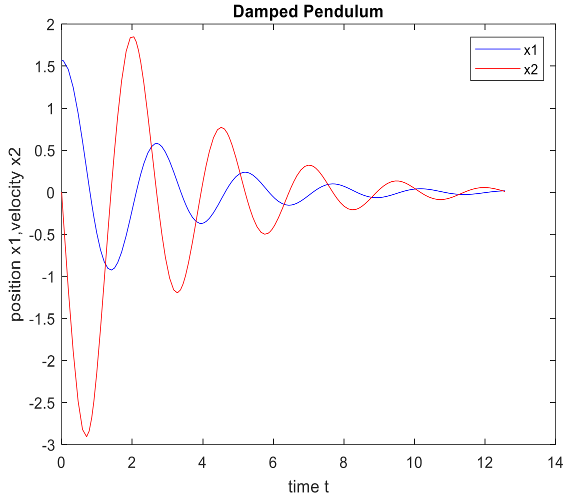

Figure 4. Damped pendulum

$$\theta(t=0) = \theta_0 = \frac{\pi}{2}\,;\, \dot{\theta}(t=0) = 0$$

This paper briefly presented the Lyapunov stability approach. The focus point was on the second method of Lyapunov. The theory has been used to investigate the stability of a simple damped pendulum. Phase portraits are shown with trajectories for some random initial conditions.

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