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A Fourier Pseudo-Spectral Method For The Benjamin Equation: Theory And Numerical Simulations

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Abstract*: We present a numerical and theoretical study of the Benjamin equation using a Fourier pseudo-spectral discretization in space. The approach is based on the discrete Fourier series and exploits spectral convergence for smooth, periodic initial data. We analyze the stability and convergence of the scheme and validate it through a series of numerical simulations. Our results demonstrate the high accuracy and stability of the method, with near-conservation of discrete invariants. A comparison was made with the existing results, and it behaves better. This work provides a foundation for further investigating dispersive nonlinear wave equations using spectral techniques.*

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I. INTRODUCTION

Evolution equations are crucial in modelling dynamic processes across various disciplines, including biology, engineering, physics, technology, and the social sciences. As a result, the qualitative and quantitative analysis of such equations constitutes a significant area of research in both pure and applied mathematics.

This study focuses on the Benjamin equation, a nonlinear dispersive model that describes the propagation of internal waves in a two-layer fluid system (Benjamin, 1996). The equation captures essential features of wave motion in stratified fluids, combining nonlinearity with nonlocal dispersive effects introduced via the Hilbert transform. Due to its rich mathematical structure and relevance to real-world phenomena, the Benjamin equation continues to attract substantial analytical and computational interest.

We consider the following form of the Benjamin equation on a periodic domain $x \in [-\ell, \ell]$:

$$\partial_t u + \alpha \mathcal{H} \partial_x^2 u + \beta \partial_x^3 u + u \partial_x u = 0 \qquad (1.1)$$

where $\mathcal{H}$ denotes the Hilbert transform, and α and β are positive constants characterizing dispersion.

The well-posedness of the associated Cauchy problem has been widely studied; see, for example, (Chen et. al., 2011; Li and Wu, 2010, Linares, 1999; Urrea, 2013) and references therein. However, obtaining explicit analytical solutions is generally infeasible due to the interplay between nonlinearity and nonlocal dispersion. Consequently, numerical methods have become essential tools for exploring the dynamics of the Benjamin equation.

Several numerical schemes have been developed for both the stationary and time-dependent versions of the problem. A key challenge in designing these methods lies in handling the nonlocal Hilbert transform, and the unbounded spatial domain when the equation is posed on the real line (Albert et. al., 1999; Calvo and Akylas, 2003; Dougalis et. al., 2015; Dougalis et. al., 2016; Kalisch and Bona, 2000).

Among the various approaches, spectral methods have proven particularly effective for periodic problems. (Kalisch and Bona, 2000) introduced a Fourier spectral method in which the Hilbert transform is treated efficiently in the frequency domain, leading to diagonal operators that simplify computation. (Albert et. al., 1999) conducted numerical simulations on vast spatial domains and obtained results consistent with earlier spectral schemes. (Calvo and Akylas, 2003) applied fourth-order finite difference discretizations to investigate interface waves governed by the Benjamin equation. (Dougalis et. al., 2015; Dougalis et. al., 2016) proposed a hybrid technique combining Fourier-type discretizations with finite-element methods and continuation algorithms. In these works, the infinite spatial domain was truncated to a finite interval with a large length L to accommodate numerical implementation. More recently, (Shindin et. al., 2021) employed a modified transmission conditions (MTC) scheme to simulate the Benjamin equation on unbounded domains directly.

In this work, we adopt a Fourier pseudo-spectral method in a periodic domain to numerically solve the Benjamin equation. This approach efficiently handles the nonlocal Hilbert term while maintaining high accuracy for smooth periodic solutions. We also compare our results with those obtained from existing numerical techniques to assess the performance and reliability of the proposed method. The structure of this paper is as follows: Section 2 covers the Fourier pseudo-spectral approximation scheme. Section 3 presents numerical simulation. Finally, Section 4 concludes the paper.

II. FOURIER PSEUDO-SPECTRAL APPROXIMATION

2.1 The periodic Benjamin equation: We apply the standard Fourier expansions to solve the Benjamin equation with the periodic boundary conditions (Benjamin, 1996; Dougalis, 2015):

$$\partial_t u + \alpha\mathcal{H}\partial_x^2 u + \beta\partial_x^3 u + u\,\partial_x u = 0, \qquad u(x,0) = u_0(x) \qquad (2.1a)$$

$$u(x,t) \text{ is } 2\ell \text{ periodic in } x, \qquad (2.1b)$$

where operator $\mathcal{H}$ is Hilbert transform, defined by the principle value integral:

$$\mathcal{H}[u](x) = \frac{p.v}{2\ell}\int_{-\ell}^{\ell} cot\left(\frac{y\ell}{2\pi}\right)(x-y)dy, \qquad x \in [-\ell, \ell],$$

The Benjamin equation is a model that describes how internal waves move in a two-fluid system. The differential equation (2.1) is conservative. Two functionals

$$\mathcal{F}[u](x) = \int_{-\ell}^{\ell}\left(\frac{1}{2}(|u|^2 - \alpha u\mathcal{H}[u_x] + \beta|u_x|^2) + \frac{1}{3}u^3\right)dx \qquad (2.2a)$$

$$\mathcal{V}[u] = \frac{1}{2}\int_{-\ell}^{\ell}|u|^2 dx \qquad (2.2b)$$

are formally preserved along with solutions to (2.1a). This can be verified by differentiating (2.2a) and (2.2b) with respect to *t* and then integrating by parts.

2.2 The numerical scheme: To solve equation (2.1) numerically, we approximate the unknown solution u(x,t) by the trigonometric polynomial $u_N(x,t)$ of degree *N* and choose unknown functions $u_{N,K}(t)$, so that differential equation (2.1a) is satisfied at the points

$$x_k = \frac{\ell(2k-N-1)}{2N+1}, \qquad k = 0,1,\dots\dots,2N+1$$

That is

$$\frac{\partial}{\partial t}u_N(x_k,t) = -\frac{\partial}{\partial x}u_N(x_k,t) - \frac{\partial}{\partial x}u_N^2(x_k,t) + \alpha\frac{\partial^2}{\partial x^2}\mathcal{H}[u_N](x,t) + \beta\frac{\partial^3}{\partial x^3}u_N(x_k,t) \qquad (2.3a)$$

$$k = 0,1,\dots\dots,2N+1$$

This gives the system of *2(N+1)* ordinary differential equations for *2(N+1)* unknown functions. Adding to (2.3a) the initial conditions

$$u_N(x_k,0) = u_0(x_k), \qquad k = 0,1,\dots\dots,2N+1 \qquad (2.3b)$$

this leads to Cauchy's initial value problem, which can be integrated numerically in time.

2.3. The numerical scheme in the modal space: It is a known fact that one-to-one correspondence between values of trigonometry polynomials at x_k and discrete Fourier coefficients. The equation (2.3) can be written explicitly using the inverse and direct Discrete Fourier Transforms**.** The formula leads to the discrete Fourier coefficients of the *m-th* order spatial derivative of $u_N(x,t)$, and are given explicitly by:

$$\left[\widetilde{\frac{\partial^m}{\partial x^m}u_N(x,t)}\right]_n = \left(\frac{i\pi n}{\ell}\right)^m \tilde{u}_n, \qquad n = -N,\dots\dots,N \qquad (2.4)$$

and

$$[\mathcal{H}[\widetilde{u_N(x,t)}]x]_n = -isgn(n)\tilde{u}_n(t), \qquad n = -N,\dots\dots,N \qquad (2.5)$$

Since there are no simple formulas for the discrete Fourier coefficients of the quadratic nonlinearity $u_N^2(x,t)$. the idea is first to compute $u_N^2(x,t)$ in physical space and then transform the results back to the Fourier space using the direct Discrete Fourier Transform formula. If we denote the discrete Fourier coefficient of $u_N^2(x,t)$ by $\tilde{v}_n(t)$, then the system (2.3) can be written explicitly as:

$$u_n'(t) = \left(\frac{i\pi n}{\ell}\right)\left[\left(1 - \alpha\frac{\pi|n|}{\ell} + \beta\frac{\pi^2 n^2}{\ell^2}\right)\tilde{u}_n(t) + \tilde{v}_n(t)\right], \quad n = -N, \dots\dots, N \qquad (2.6a)$$

with the initial conditions

$$\tilde{u}_n(0) = \frac{1}{2N+1}\sum_{k=0}^{2N} u_0(x_n)\, e^{-\frac{i\pi n}{\ell}x_n}, \qquad n = -N, \dots\dots, N \qquad (2.6b)$$

we see that the initial value problem (2.3) has a straightforward representation (2.6) in the Fourier space.

2.4. The numerical scheme in the physical space: We write the compact form of equation (2.6) as follows:

$$\widetilde{U}' = \widetilde{D}\left[\left(I - \alpha\widetilde{H} - \beta\widetilde{D}^2\right)\widetilde{U} + \widetilde{V}\right], \qquad \widetilde{U}(0) = \widetilde{U}_0 \qquad (2.7)$$

Where in modal space, $\widetilde{D}$ represents $\frac{\partial}{\partial x}$ and $\widetilde{H}$ stands for $\frac{\partial}{\partial x}\mathcal{H}$ they yield a diagonal matrix and $\widetilde{U}$ is vector matrice. We denote $U = \left(u_{N,0}, \dots, u_{N,2N}\right) \in \mathbb{C}^{2N+1}$. Let

$$F = \begin{pmatrix} e^{\frac{i\pi(-N)}{\ell}x_0} & \cdots & e^{\frac{i\pi(+N)}{\ell}x_0} \\ \vdots & \ddots & \vdots \\ e^{\frac{i\pi(-N)}{\ell}x_{2N}} & \cdots & e^{\frac{i\pi(+N)}{\ell}x_{2N}} \end{pmatrix} \in \mathbb{C}^{(2N+1)\times(2N+1)}$$

using the Fourier matrix *F* and the inverse and direct discrete Fourier transform, then

$$U = F\widetilde{U}, \qquad \widetilde{U} = \frac{1}{2N+1}F^*U \qquad (2.8)$$

$*$ mean conjugation, since formula (2.8) holds for any vector $U \in \mathbb{C}^{2N+1}$, then $F^{-1} = \frac{1}{2N+1}F^*$.

This property gives us leverage to rewrite equation (2.7) in terms of vector U:

$$U' = D[(I - \alpha H - \beta D^2)U + U^2], \qquad U(0) = U_0 \qquad (2.9)$$

The Fourier differentiation matrix D and its coefficient can be computed explicitly using

$$d_{i,j} = \begin{cases} \dfrac{(N+1)sin\left(\frac{2\pi(i-j)}{2N+1}N\right) - Nsin\left(\frac{2\pi(i-j)}{2N+1}(N+1)\right)}{2\ell(2N+1)sin^2\left(\frac{\pi(i-j)}{2N+1}\right)}, & i \neq j \\ 0, & i = j \end{cases} \qquad (2.10)$$

where $0 \le i, j \le 2N$.

2.5. Stability: We know that the exact solution of equation (2.1) is conservative, i.e

along the trajectories of (2.1), two first integrals $\mathcal{F}[u]$ and $\mathcal{V}[u]$ are exactly preserved. In particular, the integral $\mathcal{V}[u]$ shows that the $L_P^2(-\ell, \ell)$ norm of u remains bounded for all times $t \ge 0$. In general, a numerical discretization does not have to be automatically conservative. Fortunately, in the case of the discretization (2.3), the numerical solution has two discrete first integrals that correspond to $\mathcal{F}$and $\mathcal{V}$.

Theorem 2.1. The approximation solution $u_N(x,t)$ satisfies:

$$\mathcal{F}_N[u_N](t) = \frac{2\ell}{2N+1}\sum_{n=0}^{2N}\left(\frac{1}{2}\left(u_N^2(x_n) - \alpha u(x_n)\mathcal{H}[u_{Nx}](x_n) + \beta u_{Nx}^2(x_n)\right) + \frac{1}{3}u_N^3(x_n)\right)$$
$$= \mathcal{F}_N[u_N](0), \qquad t \ge 0, \qquad (2.11a)$$

$$\mathcal{V}_N[u_N](t) = \frac{\ell}{2N+1}\sum_{n=0}^{2N} u_N^2(x_n) = \mathcal{V}_N[u_N](0), \qquad t \ge 0, \qquad (2.11b)$$

Proof: We omit the proof.

Formulas (2.11) are significant because they demonstrate that the numerical solution preserves key qualitative features of the exact solution. In particular, formula (2.11b) shows that the discrete $L_P^2(-\ell, \ell)$ norm of the numerical solution remains invariant over time. This implies that the solution does not exhibit unbounded growth as *t* increases, thereby confirming the stability of the numerical scheme (2.3).

III. NUMERICAL SIMULATION

We now present several numerical examples to validate the accuracy and stability of the proposed method.

Example 1: Exact Solution with Cosecant Initial Data

We introduce a source term such that the modified equation admits the exact solution $u(x,t) = e^{-t}cosec\left(\frac{\ell}{2\pi}x\right)$. The numerical solution using *N=65* collocation points is compared to the exact solution over $t \in [0,5]$. The error remains below 10^{-5} throughout the domain.

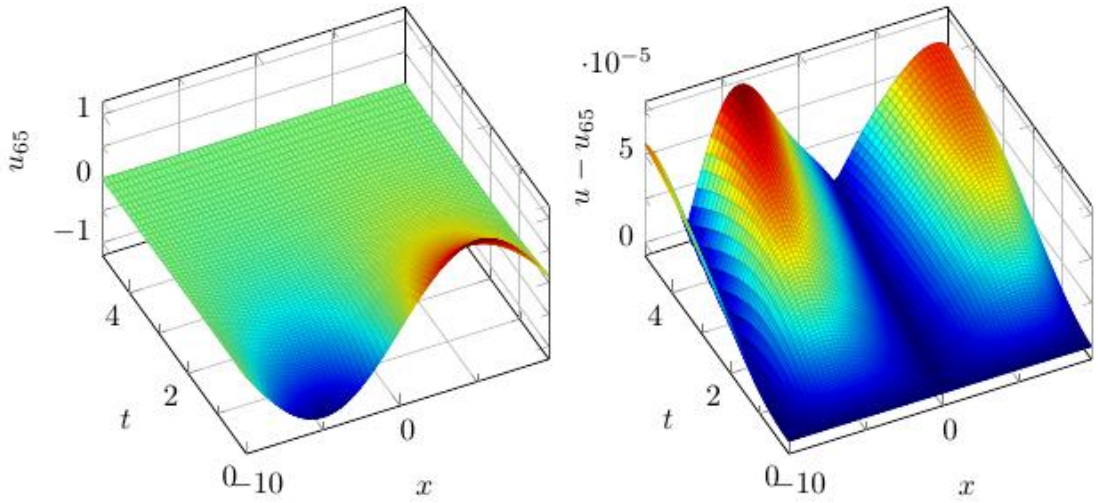

Figure 1: Numerical solution (left) and pointwise error (right), $\propto, \beta = 1$.

Example 2: Exact Solution with Secant Initial Data
Using a similar setup with $u(x,t) = e^{-t}sec\left(\frac{\ell}{2\pi}x\right)$. The numerical scheme again produces a stable solution, with errors of order 10^{-4}.

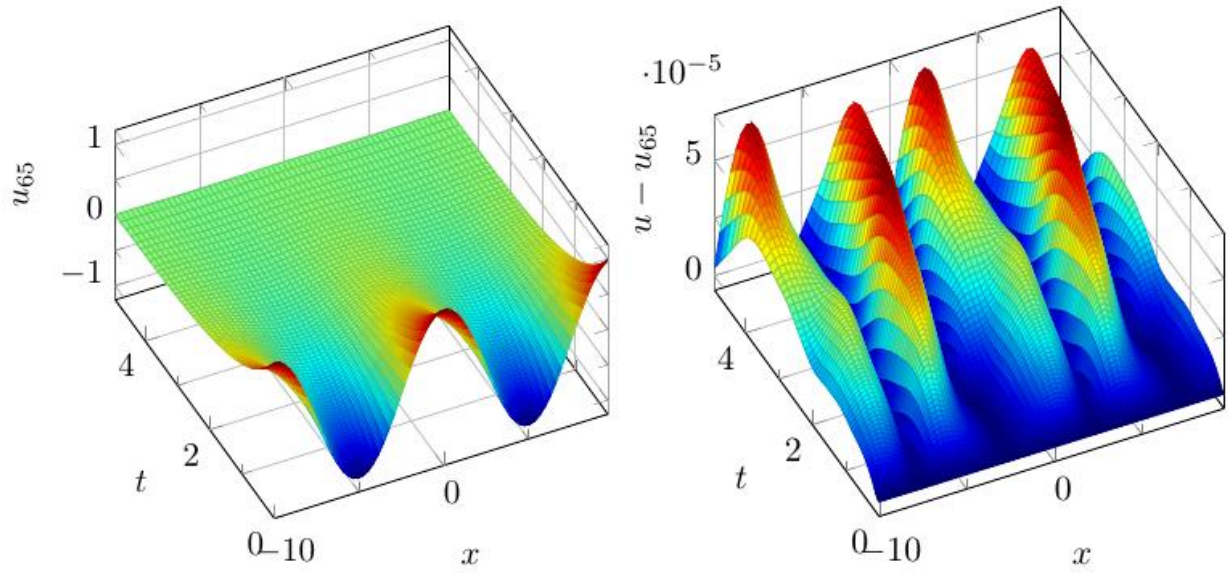

Figure 2: Numerical solution (left) and pointwise error (right).

Example 3: Solitary Wave and Conservation Testing
We initialize the solution with a known solitary wave profile from the KdV equation and observe the behaviour under the Benjamin equation. The invariants $\mathcal{F}_N$ and $\mathcal{V}_N$ exhibit near-conservation despite non-conservative time-stepping.

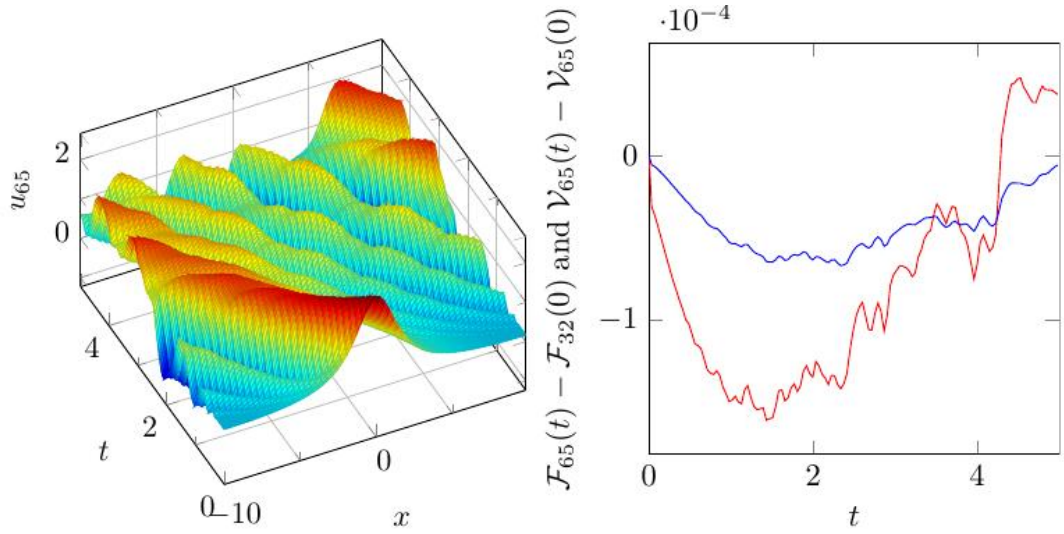

Figure 3: Numerical solution (left) and deviations in invariants (right).

Example 4. Comparison with Finite Difference Method

To benchmark our pseudo-spectral scheme, we implemented a second-order finite difference (FD) discretization of the Benjamin equation with central differencing for spatial derivatives and RK4 in time. For the test case in Example 1, the FD method required *N=512* grid points to achieve comparable accuracy (error 10^{-5}) to our spectral scheme using *N=65*.

Method	Grid Points	Max Error
Pseudo-spectral	$N = 65$	1.6×10^{-5}
Finite Difference	$N = 512$	1.3×10^{-5}

Table1: Comparison of accuracy for Example 1 between pseudo-spectral and FD methods.

IV. CONCLUSION

We have presented a Fourier pseudo-spectral method for solving the Benjamin equation in a periodic domain. The method leverages spectral accuracy for smooth solutions and exhibits favourable conservation and stability properties. Numerical experiments validate the theoretical predictions and show excellent agreement with known solutions. Future extensions may include conservative time integrators, adaptation to non-periodic settings, and applications to more general nonlinear dispersive models.

We developed and tested a Fourier pseudo-spectral method for numerically solving the Benjamin equation under periodic boundary conditions. The scheme, expressed in both modal and physical space, was analyzed for accuracy and stability. A split formulation using an integrating factor enhances time integration stability. Comparisons with finite difference methods show spectral methods achieve high accuracy with fewer grid points. Our simulations demonstrate the method's ability to conserve key invariants, resolve solitary waves, and maintain long-time accuracy. This confirms its suitability for efficient and reliable simulation of dispersive nonlinear equations such as the Benjamin equation.

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