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Why Asymptotic Behaviour of Limits is Important for Students to Learn before Differentiation ?

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Abstract
This paper explores how limits can be taught in an intuitive and meaningful way. Understanding the asymptotic behaviour of limits is a fundamental step in preparing students for differentiation in calculus. Limits describe how functions behave as inputs approach specific values or infinity, offering insight into long-term trends that are essential for grasping rates of change [1][2].Before learning differentiation, students must understand how functions grow, decay, or stabilize, as these behaviours directly influence core calculus ideas such as continuity, slopes, and instantaneous rates of change [25]. Without this foundational understanding, the process of differentiation can feel abstract and disconnected from real-world applications [27].Moreover, asymptotic analysis helps distinguish between functions that grow at different rates—such as exponential, logarithmic, and polynomial functions—which is essential for interpreting derivatives effectively [25][26]. Therefore, studying the asymptotic behaviour of limits equips students with both mathematical tools and conceptual intuition, enabling them to understand differentiation more deeply and apply it accurately [2][26].
Keywords: *Limits, Asymptotic behaviour, Differentiation Calculus, Education, Intuitive Learning, Conceptual Understanding, Rates of Change, Infinity in Mathematics, Graphical Learning, Mathematical Intuition, Precalculus Foundations.*

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1.1.1 Why it is important to Look at the concept of Limits from Different Perspectives?

The mathematical community has long accepted that limits are the foundation of calculus, serving as the first essential concept for students to grasp in their study of the subject. However , over time, the importance of limits has been overshadowed, and due to the way curricula are structured, the topic is often rushed through without adequate focus. In many cases, limits have become merely a procedural topic, tested for solving problems but not explored in depth for their conceptual understanding. As a student myself, I initially thought I fully understood limits because I scored 100% in my exams. However, after taking a deeper five, I realized that I didn't truly grasp what a limit represents.

Even though students are taught how to solve limits, the deeper understanding of how limits actually work is often neglected. For example, in Chapter 2.4 of Early Transcendentals, we are introduced to the formal definition of a limit, which is essential for developing an intuitive grasp of the concept. However, this chapter was not part of the official syllabus, as it was not directly focused on solving problems. As a result, it was often overlooked in favour of more problem-oriented content. Despite its importance in building a conceptual foundation for limits, many students didn't engage with this chapter because it was not considered essential for exams. Only those students who were willing to go beyond the basic requirements and invest extra effort in understanding the formal definition of a limit were able to gain a deeper comprehension of the subject – and these were often the students who excelled in exams [1].

To develop a true understanding of limits, it's essential to examine them from different perspectives. Graphically, limits can be visualized to show how a function behaves as it approaches a specific value or infinity. This approach develops intuitive understand of limits and this skill also helps in future more complex concepts. As Liang (2016) discusses, visualizing limits graphically allows student to gain an intuitive understanding of how functions behave as they approach certain values, without needing to rely solely on algebraic calculations [3]. Graphing helps students to see the behaviour of functions in ways that numeric or algebraic representations alone cannot . This method makes abstract concepts like infinity and discontinuities more concrete and accessible.

Studies support the idea that graphical representations of limits improve students' comprehension

[4][5][6][7]. Additionally, graphing provides valuable insight into the behaviour of functions near critical points, making it easier for students to recognize asymptotes, continuity, and the approach of values without needing to perform complex calculations [8]. When students interact with graphs, especially using software that allows dynamic exploration of functions, they develop the ability to understand limits not just as a symbolic concept, but as a dynamic process [9].

Ultimately, the primary goal of introducing limits before differentiation and other calculus concepts is to encourage students to approach these topics with an intuitive understanding. When students grasp the underlying behaviour of functions through the concept of limits, they develop a deeper interest in the subject. This foundation allows them to confidently tackle more complex problems. Furthermore, when faced with challenging questions, students with an intuitive grasp of limits can apply this understanding to navigate and solve tough problems, even when the solution is not immediately apparent.

1.1.2 Understanding the Idea of Infinity

The idea of infinity is something I still feel I have incomplete knowledge about but over the past 10 years that I have been studying about infinity and concepts revolving around infinity and I have a definition for infinity in my own terms. According to many philosophers and mathematicians concede that number are not real or are abstract, but these is a disagreement weather numbers exit independently of rational observers [10].For this paper and in the field of mathematics infinity has been defined by the German mathematician George Cantor (1845-1918) [11]. For learning asymptotic behaviour learning the idea of infinity is important and that is why I have a definition that I fell does justice to the idea of limits.

Definition of Infinity – Infinity is a concept rather than a specific number. It describes the behaviour of a quantity that grows without bound or becomes arbitrarily large. Infinity is not a fixed value or a point on the number line; rather, it is an abstract idea used to represent unbounded growth. In mathematics, we say that a function approaches infinity (or tends to infinity) when its values increase beyond any finite limit, and this process can be studied by examining the rate or speed at which the function grows. While we cannot calculate infinity itself, we can analyse how a function behaves as it "approaches infinity" – that is, as it grows larger and larger without ever reaching a definitive , calculable value.

If you read the definition I say that "Infinity is not a fixed value or a point on the number line" but it actually can be and when we are solving question on asymptotic behaviour or thinking about infinity in the concept of asymptotic behaviour we should see infinity as a symbolic point that represents the extreme ends of a function's growth or decay . While infinity is not a specific number or a finite value, it is used in limits to describe the behaviour of functions as they approach unbounded growth (positive infinity) or decay (negative infinity).

In asymptotic analysis, we often treat infinity as a point on the extended real number line. This extended number line includes both $+\infty$ and $-\infty$ as points at the "ends" of the number line, allowing us to describe the behaviour of functions as they grow without bound or decrease infinitely. While these points are not actual numbers that can be operated on arithmetically, they serve as a useful tool in understanding the limits and rates of growth of functions [1][2].

Another Idea that I absolute love is Torricelli's Trumpet [12][13]. The idea of the horn is so insightful . It is close related to one of the examples described further in the paper. In simple terms if we turn around the graph of y=1/x around the x axis we get a flared bell trumpet. Also known as the Gabriel's horn, in reference to the biblical Archangel Gabriel. The idea was formed as the horn as infinite surface but finite volume . Through this if x tends to infinity then the horn's surface area also tends to infinity, while the volume enclosed by the curve tends towards the value of pi cubic units (where pi = 3.1416…) . The question still remains that why a finite volume is bound by an infinite area .

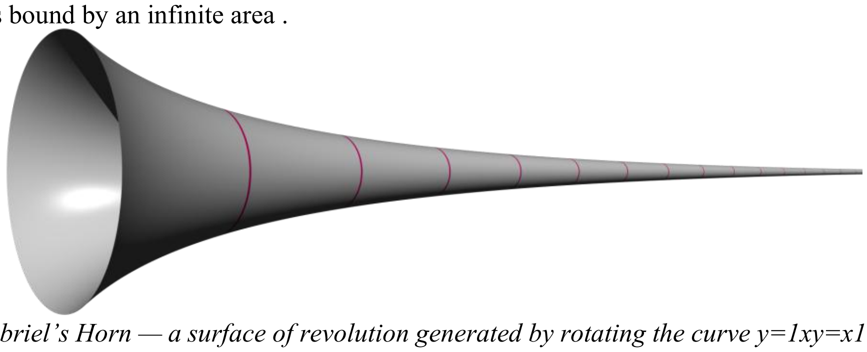


Figure 1. Gabriel's Horn — a surface of revolution generated by rotating the curve y=1xy=x1 (for x≥1x≥1) about the x-axis. It has finite volume but infinite surface area.

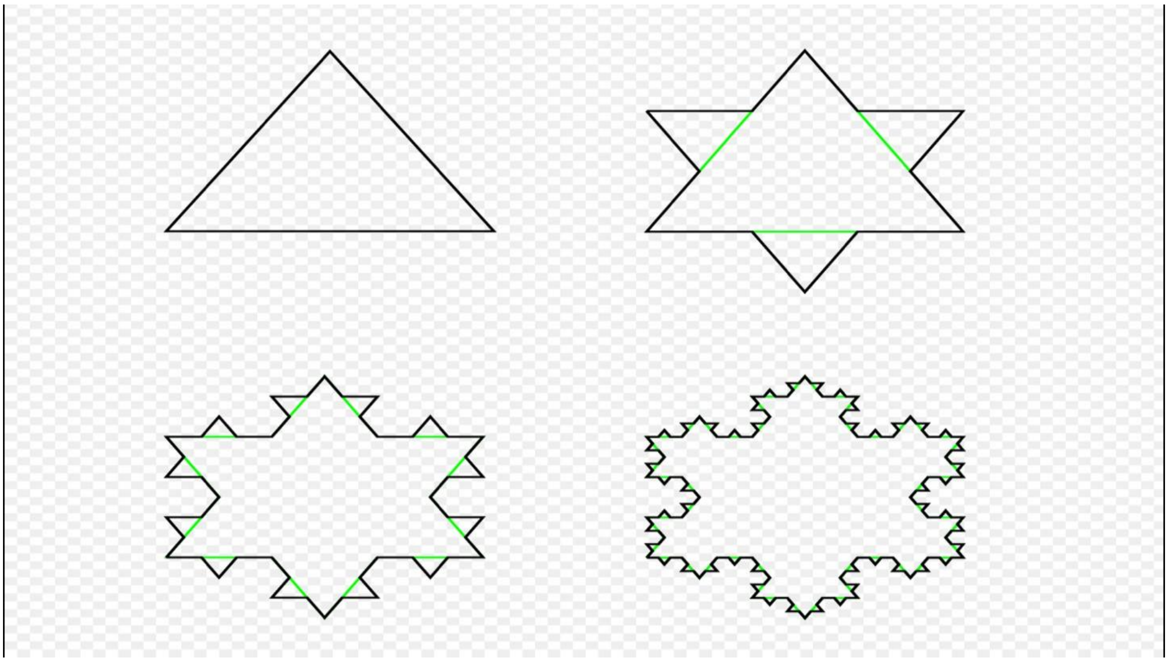
Source: [14]

Introducing infinity to students before delving into the concept of limits is a powerful and effective way to spark their interest and help them understand infinity as a multifaceted concept. Often, infinity is taught only as part of the formal study of limits, which can lead to confusion and misconceptions. The lack of foundation further huts the students idea of limits , causing them to struggle with more advanced topics later. Developing the idea of infinity with real world examples like Torricelli's Trumpet can be helping in making students gain institutive understanding of the concept and this skill can be further used in the topic of asymptotic behaviour [14].

1.1.3 Koch snowflake Fractals

Another way you can understand the idea of infinity is through the Koch snowflake which is a famous fractal curve and one of the most well-known examples of self- similar geometric figure . The idea of infinity can be understood from this too. Imagine starting with an equilateral triangle. This is how the shape initially starts and then we have to divide the triangle into three equal parts and then construct an equilateral triangle that has the middle segment of each side as base .After that remove the middle segment of the original triangle .Know the shape has become 12 sided and as you continue the process the snowflake keeps on glowing to 48 then to 192 to then you can keep on going infinitely .

The interesting thing is that the number of iterations tend to infinity . The interesting idea is the infinity in time where you can never reach all iterations and this tells use how this is a never ending process. As you keep on adding more and more triangle the perimeter of the Koch snowflake increase infinitely .The idea is institutive as if you wanted to fence the snowflake you would never be able to do it even though the shape remains finite in other aspects. So even though the shape has finite area the perimeter is infinite. Due to this paradox we can see the different magnitudes of infinity [15][16] .



Koch Snowflake — a fractal curve formed by iteratively adding equilateral triangles to each side of a triangle. ***.*** *The process leads to a shape with an infinite perimeter but a finite area, illustrating the counterintuitive properties of geometric infinity.*
Source**:** [35]

This distinction aligns with concepts developed by mathematician George Cantor, who showed that not all infinities are equal. While the number of iterations in constructing the snowflake tends to infinity, the area enclosed by the shape remains bounded, which offers a tangible way to grasp the counterintuitive properties of infinite processes [11][12].

The idea of the snowflake helps to see how self – similarity works .The idea of this is used by people to make

antennas . Due to the unique geometric properties of the Koch snowflake, antennas designed with fractal structures can operate efficiently at multiple frequencies. This is because the self-similarity of the fractal shape allows for the reception and transmission of electromagnetic waves across different scales, making such antennas particularly useful in telecommunications and signal processing [15]. This parallels fundamental concepts in calculus, where function approach specific values as inputs tend to infinity [1][2]. The Koch snowflake serves as an example of how infinitely small changes, when repeated endlessly, can lead to well-defined mathematical outcomes, reinforcing the importance of limits in understanding continuous and discrete processes [3][4].
Unlike traditional geometric shapes that have integer dimensions, fractals like the Koch snowflake exhibit non-integer, or fractional, dimensions. This idea was first formalized by Benoit B Mandelbrot, who introduced the concept of fractal dimension to describe the complexity of self-similar structures [16]. The Koch snowflake, for instance, has a fractal dimension of approximately 1.2619, which places it between a one-dimensional line and a two-dimensional plane [15][16]. This challenges conventional notions of geometry and highlights how infinity manifests not only in size but also in the very nature of space and form.

Ultimately, the Koch snowflake serves as a powerful tool for visualizing and understanding infinity. It presents an intuitive example of how infinite processes can exist within a finite space, offering propound implication in the studying of infinity and ultimately limits [12][8]. Understanding the idea of infinity is important for limits. It allows the conceptual foundation for limits by allowing us to think about quantities that grow without bound or approach an elusive value[1][3]. In many cases, we analyse how a function behaves as its input becomes infinitely large, which helps us determine asymptotic behaviour – how a function stabilizes, diverges, or oscillates at extreme values [1][2]. Similarly, when dealing with infinitely small quantities, we use limits to define concepts like continuity and derivatives, which require understanding how a value can get arbitrarily close to another without necessarily reaching it [3][8].

On the other hand limits allow us to control and make sense of infinity by describing what happens at the “edge” of infinite processes. For example, when a function approaches a certain value as its input grows infinitely large, the limit provides a precise way to define that tendency without requiring an actual infinite calculation. This is crucial because infinity itself is not a number but a concept, and limits provide the mathematical structure to handle it in a rigorous way [1][8].

By understanding infinity, we recognize that some processes never truly “end” but still settle towards predictable outcomes. This insight is crucial in calculus, analysis, and real-world applications where we deal with infinite sequences, rates of change, and summations. This, infinity is not just an abstract notion – it is the key to understanding how limits allows us to describe and predict the behaviour of functions in both mathematical theory and practical applications [1][3].

1.2 Importance of learning asymptotic behaviour to understand the bigger picture

While studying a new concept the key to gaining interest in the concept is to understand it in the real world as this application helps the student get more connected to what he or she is studying [3][5]. Leaning of asymptotic behaviour has know become just a way to solve questions that tend to infinity but I feel it is much more that that. In the long run understanding limits that tend to infinity can be beneficial to understand complex concepts like linear approximation or Taylors Theorem [1][2].

Beyond just solving problems, asymptotic behaviour provides a deeper insight into how functions behave at extreme values, revealing patterns that might not be obvious at smaller scales [8]. It helps in understanding efficiency in computer science, where algorithm complexity is analysed using Big-O notation, ensuring that systems can handle large inputs efficiently [7][17]. In physics, asymptotic analysis helps in studying wave behaviour, relativity, and quantum mechanics, where values approach limits that define fundamental constants [8][18]. Engineering applications, such as signal processing and control systems, also rely on asymptotic understanding to optimize performance and stability [6][19]. Thus, learning asymptotic behaviour is not just about solving mathematical limits but also about how mathematical structures govern real-world phenomena [3][9].

As a student I wanted to design a few problems that can be taught in class just to give students enough knowledge so that their interest is developed and they have enough reason to understand why they need to understand limits. One the goals of this paper is to make a path that a professor can follow in order to make the students interested in this topic [4]. Often professors have a lot more knowledge that some random student like me but the one thing he might miss is how a student perceives and digests information [5]. In order to get through student the professor has to connect the topic the to real world problems and also other fields [3][6].

This will help the students that are taking Math courses even though their major is not Maths. The majority of the class taking lower division math courses are taking the class due to this requirement[3].

To cater to the majority of the class the concept of limits has to be taught in a way that the students will be able to see the importance of it in their own field [7]. That is why at the end of each question I have also added a small section that only speaks about how the question is related to the real world . This would provide them with the incentive on focus on the topic more. In addition to this the students can have a small section in examination in which they can explain how the understand of limits can apply to a question. I will be providing 5 original question that are a guide on how a student can be tested on the intuitive understand of limits [4][7].

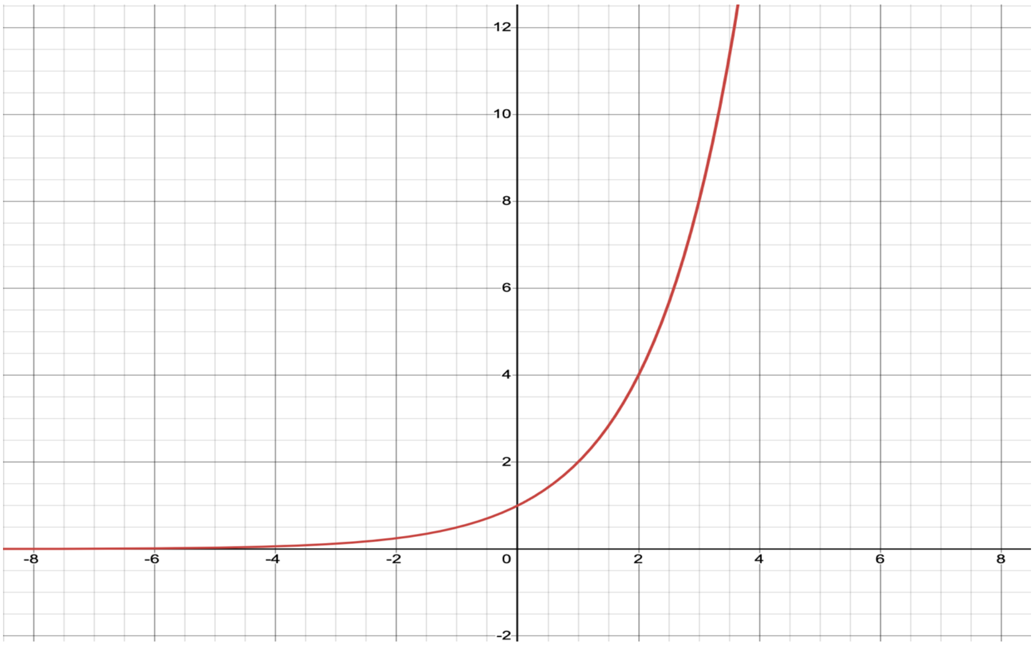
Example 1 : Exponential Growth
Concept Covered : Exponential Growth, Asymptotic Behaviour, Limits

Consider the function $f(x) = 2^x$, which represents exponential growth. Question: What happened to the value of 2^x as $x \to \infty$?
Question: What happens to the value of 2^x as $x \to -\infty$?

When we examine the $\lim x \to \infty\ 2^x = \infty$, we are focusing on the asymptotic behaviour of the function as x becomes very large.

Mathematically, we can write:
$\lim x \to \infty\ 2^x = \infty$
This is the basic knowledge of asymptotic behaviour and so know we have to dive deeper into this question. The rate of growth is curtail here. Since the base (2) is greater that 1, the function grows exponentially, meaning it increases very rapidly as x increases [27]. The behaviour is important for exponential growth.

If we compare 2^x to a polynomial function like x^2, exponential functions eventually outgrow polynomials for large x, even though both functions increases as $x \to \infty$. The term asymptotic refers to this “infinite” growth: as x gets larger and larger, the function behaves in a way that approaches infinity. We notice that the function never levels off, but continues increasing faster and faster [23].



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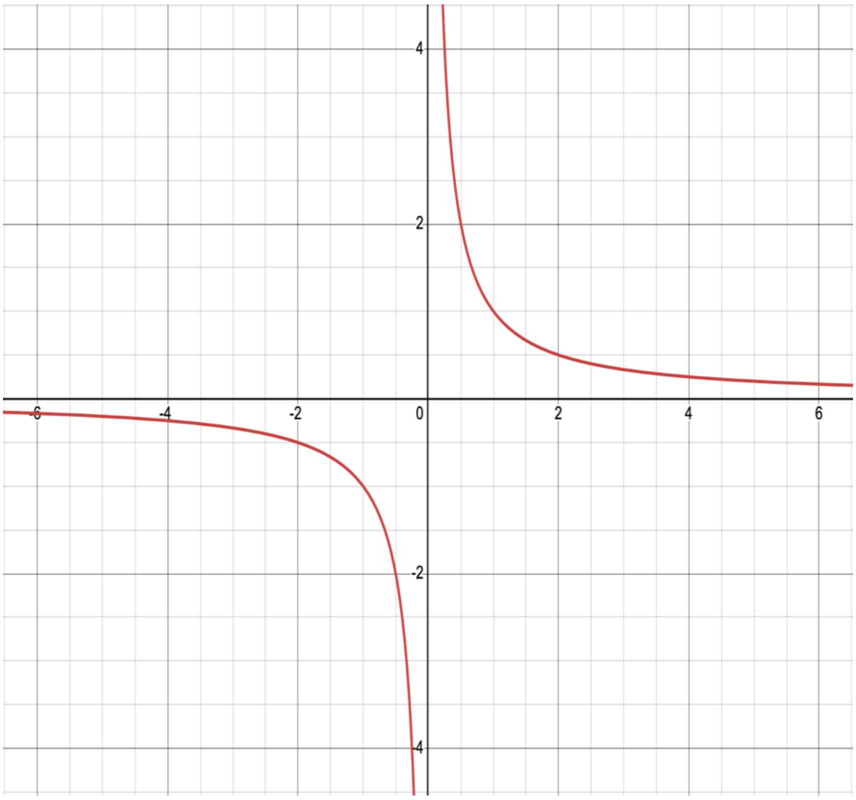
As $x \to -\infty$, 2^x decreases towards 0 but never actually reaches it. This is because $2^{-1} = 1/2$, $2^{-2} = 1/4$, $2^{-3} = 1/8$, ……
As x becomes more negative , 2^x keeps getting smaller, approaching zero. Mathematically, we can write:
$\lim x \to -\infty\ 2^x = 0$
In this case, we can describe the function decaying asymptotically towards zero. It never quite reaches zero, but it gets arbitrarily close to it as x becomes increasingly negative. This type of behaviour is known as asymptotic

behaviour, where the function gets closer and closer to a value (in this case zero) but never actually reaches it. In this case, 0 is the horizontal asymptote of the function as x tends to negative infinity [1][2][24].
The question is a very nice insight to real world problems like population dynamics and introducing the problem in that way would engage students in the topic and help the student see the importance of asymptotic behaviour. The fast growth of the function can also have application in the understanding of orders of growth which is important in computer science and algorithm design [27].

Example 2 : Inverse Proportionality
Concept Covered : Asymptotic Behaviour, Limits

Consider the function $f(x) = 1/x$
Question: What happened to the value of $1/x$ as $x \to \infty$? Question: What happens to the value of $1/x$ as $x \to -\infty$?
When we examine the $\lim x \to \infty\ 1/x = \infty$, we are focusing on the asymptotic behaviour of the function as x becomes very large [1][2].
Mathematically, we can write:
$\lim x \to \infty\ 1/x = 0$

For this question when x approaches infinity, or in other words x becomes a very large number or grows without any bounds. Due to this reason the value gets closer and closer to 0 but never reaches it . This is how I thought of the question when I first solver and ironically the answer is also correct but there is a big misconception which me and many student have when they solve this kind of a question. As earlier stated the limit does not reach to 0; in calculus
, when we say the $\lim x \to \infty\ 1/x = 0$, we mean that $1/x$ becomes arbitrarily close to 0 as x grows larger [27] . Know when x approaches negative infinity, or in other words x becomes a very large negative number and continues to grow without bound in the negative direction, the value of $1/x$ gets closer and closer to 0 similar to when x is tending to positive infinity . The only difference is that the function is tending to negative infinity and so is coming closer and closer to the value of 0 from the other side [23] .
The limit describes the behaviour but not the actual value of f(x). The phrase the " never reaches to 0" might suggest that f(x) is trying to reach 0 but failing to do so. This is misleading because f(x) isn't "struggling" to reach any particular value. Instead when x tends to infinity f(x) simply becomes smaller and smaller , indefinitely approaching
0. A better way to phrase this to students is to say that the function can approach zero "infinitely" as x increases.



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The other misconception that this question clears is that asymptotic limits deals with infinity rather than exact points

. In other words infinity should be treated as direction and not as a destination . In calculus, when we work with limits approaching infinity, we're interested in how a function behaves as the input value grows larger and larger, not at a specific value is "reached" at some point [1][2]. Another interesting thing that can be seen in this question is how rate of decay slows down as x increases. The idea of rates of change should be introduced through this question as it will see a easier path for differentiation [27]. The question has application in real world problems too. The students should be introduced to this question as an inverse square law question. It also serves as a nice example for resistance in electrical circuits . This is a nice way to make the question interesting [20][21] .

Example 3 : Exponential Decay

Concept Covered : Exponential Decay, Asymptotic Behaviour, Limits

Consider the function f(x) = e^(−x)

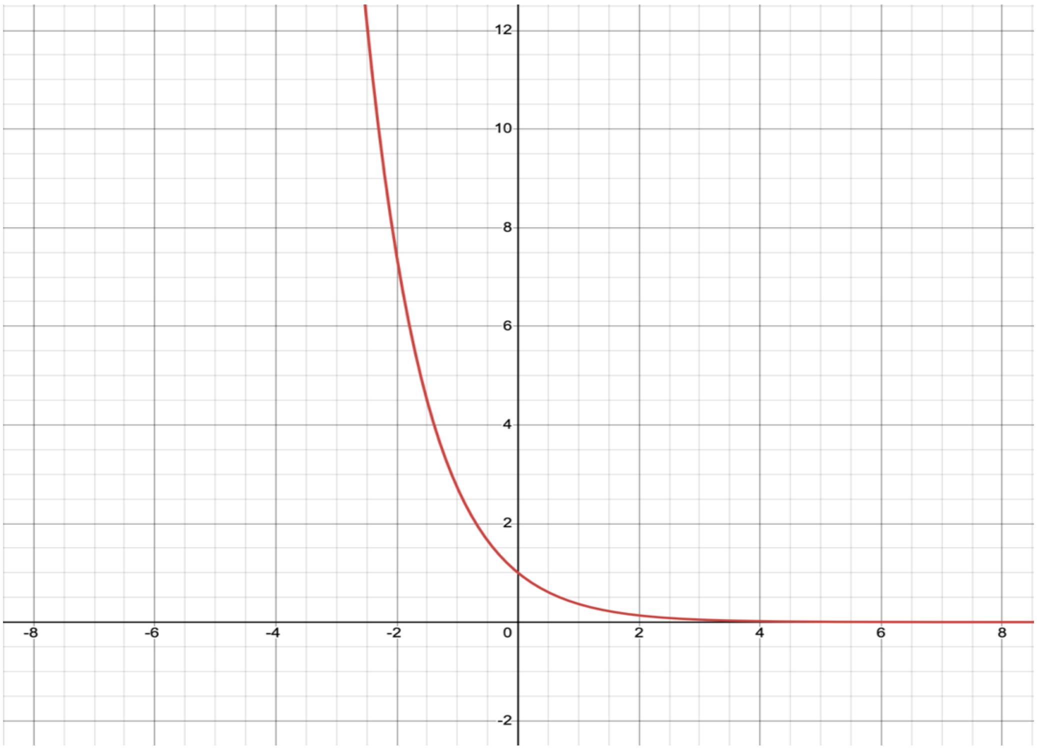
Question: What happened to the value of e^(−x) as x → ∞? Question: What happens to the value of e^(−x) as x → -∞?

When we examine the lim x → ∞ e^(−x) = ∞, we are focusing on the asymptotic behaviour of the function as x becomes very large [1].

Mathematically, we can write:

lim x → ∞ e^(−x) = 0

This means that as x becomes very larger, the functions decays towards to zero. The term asymptotic decay describes this behaviour of e^(−x) approaches zero but never actually reaches it . This question is exactly opposite of the first question . Another way to look at this question which we did not use in the first example is to plug in the values of x to see what happens . For example if we plug in -10 in place of x the value becomes 0.000045 and then if we put -20 in place of x then it becomes 0.000000002.



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This shows as x increases the value of $e^{(-x)}$ gets smaller and smaller, approaching zero. This is the basic knowledge of asymptotic behaviour and so know we have to dive deeper into this question. The base (approximately 2.718) is greater than 1. When the exponent is negative, it leads to division, causing the function to shrink exponential.

In the case when x tends to negative infinity the function grows exponentially. To solve this we can rewrite $e^{(-x)}$
as $1/e^x$. When x tends to negative infinity the function become equivalent to $e^{(|x|)}$ where |x| is very large.
Mathematically, we can write:
$\lim x \to -\infty\ e^{(-x)} = +\infty$
This means that as x becomes more negative as $e^{(-x)}$ grows arbitrarily large. We can again repeat and take values to further check our answer by taking values. For example if we plug 10 in place of x the value becomes 22026.465 and then if we put 20 in place of x then it becomes 485165195. The reason I chose this question is because, unlike other functions of the form e^x, which typically grow for positive values of x and decay for negative values, this function exhibits growth as x tends to negative infinity [27]. This behaviour is the opposite of what you'd expect from a standard exponential decay function [8].

The question also has real world application such as radioactive decay and similar to the first question it also has applications on population dynamics . The decay of radioactive substances is modelled using exponential functions and exponential decay plays an important role in it. Similar to the first example this question also has implications on population dynamics. The function plays a major role in Newton's Law of Cooling and how an object cools over time [20][21].

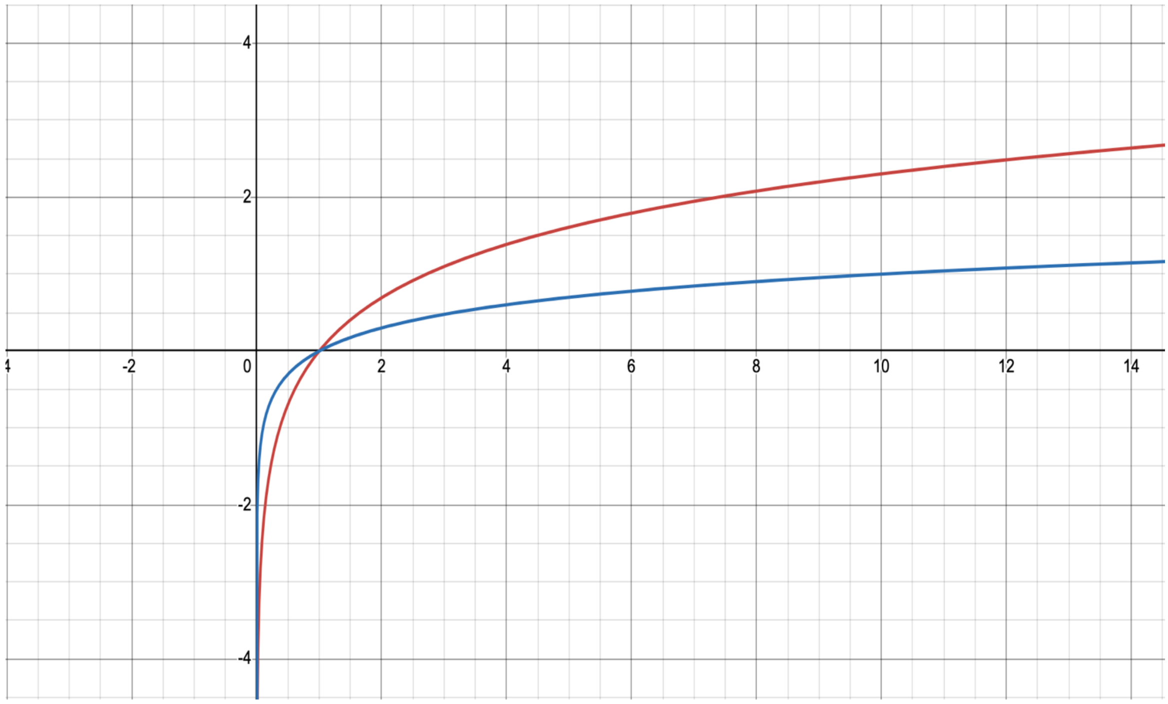
Example 4 : Logarithmic Growth
Concept Covered : Exponential Decay, Asymptotic Behaviour, Limits

Consider the function f(x) = ln(x) [natural logarithm] and log(x) [logarithm base 10] Question: What happened to the value of ln(x) and log(x) as $x \to \infty$?
Question: What happens to the value of ln(x) and log(x) as $x \to -\infty$?

When we examine the ln(x) and log(x) , we are focusing on the asymptotic behaviour of the function as x tends to positive and negative infinity [1][2].

Mathematically, we can write:
$\lim x \to \infty \ln(x) = \infty$
As x increases without bound, ln(x) continues to grow, but its growth rate slows significantly [8]. This means that ln(x) is unbounded, but the rate which it gets big decreases for larger values of x.

Mathematically, we can write:
$\lim x \to \infty \log(x) = \infty$
Similarly, log(x) {blue line} also grows indefinitely, but even slower than ln(x) {red line}. The reason for this is because log(x) is scaled by ln(10) [23]. This makes the function grow but compared to the rate of ln(x) it grow even slower but similar to ln(x) the log(x) continues to grow [7]. While both ln(x) and log(x) functions grow when x tends to infinity , they grow much slower that linear or exponential functions. The key misconception that students might have when looking at logarithmic function could be that logarithmic functions are "approaching infinity more slowly" [9].



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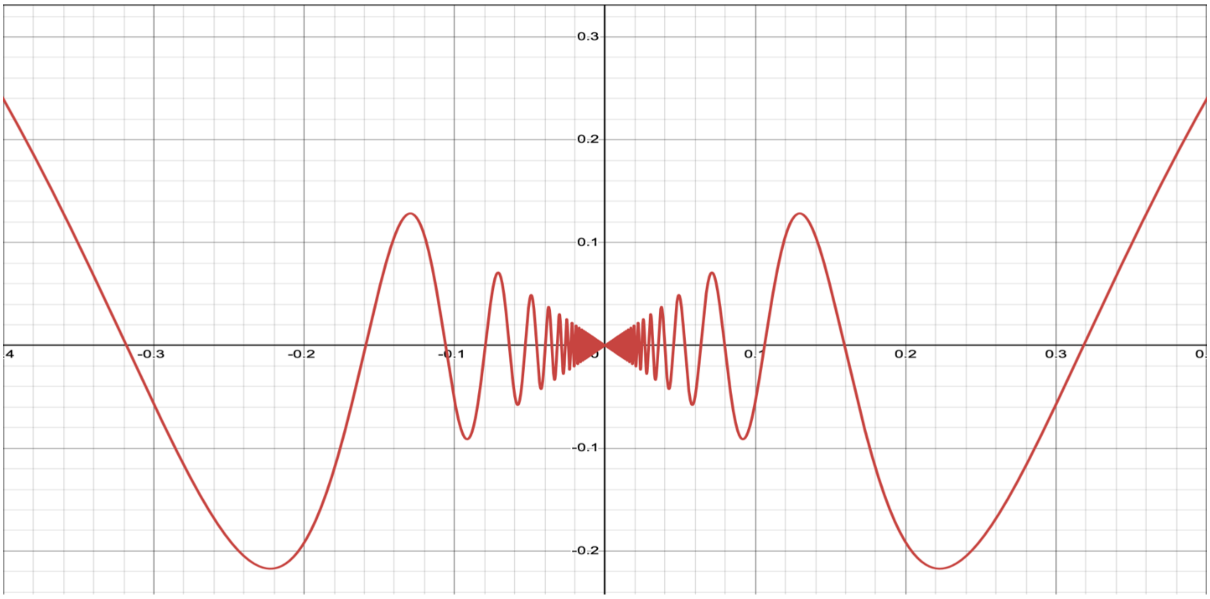
A better way to make students clear this misconception is it say that the logarithmic functions have an unbounded growth with a decreasing rate of growth . Having basic concepts like these clear in a students mind would help them grasp concepts of differentiation quicker [9].
The other misconception that students may have is that logarithmic functions, such as ln(x) and log(x), will eventually "level off" or reach a limit as x increases, i.e., that they might approach a finite value [1][7]. This is however not true but as I mentioned earlier that these function grow continuously without any bounds but the rate at which it grows slows down significantly [8]. An interesting way to phrase this by saying that logarithmic functions always increase just at a decreasing rate . This distinctions is very important compared to linear or exponential ones as they grow at a much faster rate [9].

In real life logarithmic growth often is seen in population growth, information theory(e.g., data compression or entropy), and the measurement of sound intensity (decibels) .In the field of information theory, the amount of information needed to describe a system might grow logarithmically but in accordance to the number of elements in the system [20][4]. For differentiation understanding the concept of logarithms is very important as the derivative of ln(x), for example, is $1/x$, which reveals that the rate of change of the logarithmic function decreases as x increases. To interpret the behaviour of a function it is important to understand the idea of the decreasing rate of change of the logarithmic function [21].
Another prospective to look at logarithmic functions are through concavity . Even though I have not mentioned concavity that much but I feel it is important to understand how concavity relates
to the nature of logarithmic functions. The second derivative of ln(x) is given by $-1/x^2$, which is always negative for positive x. this means that the function is always concave down, indicating that its rate of growth is perpetually decreasing. This concavity explains why logarithmic functions never grow as quickly as polynomial or exponential functions and why they are useful in modelling diminishing returns in real-world applications [10].
By understanding concavity, students can also see why logarithmic functions serve as good models for processes where initial rapid growth slows over time, such as learning curves or economic scaling laws [7][8]. This deepens the connection between limits, differentiation, and real-world applications, reinforcing the importance of conceptual clarity when studying the concept of limits [10].

Example 5 : Oscillatory Function
Concept Covered : Asymptotic Behaviour, Limits

Consider the function $f(x) = x \sin 1/x$, $x \neq 0$. Question 1: What happens to f(x) as $x \to 0^+$?
Question: What happened to the value of $x \sin 1/x$ as $x \to \infty$? Question: What happens to the value of $x \sin 1/x$ as $x \to -\infty$?

When we examine the function x sin $1/x$, we are focusing on the asymptotic behaviour of the function as x tends to positive and negative infinity [6][8].
Mathematically, we can write:
$\lim x \to +\infty \; x \sin 1/x = 0$



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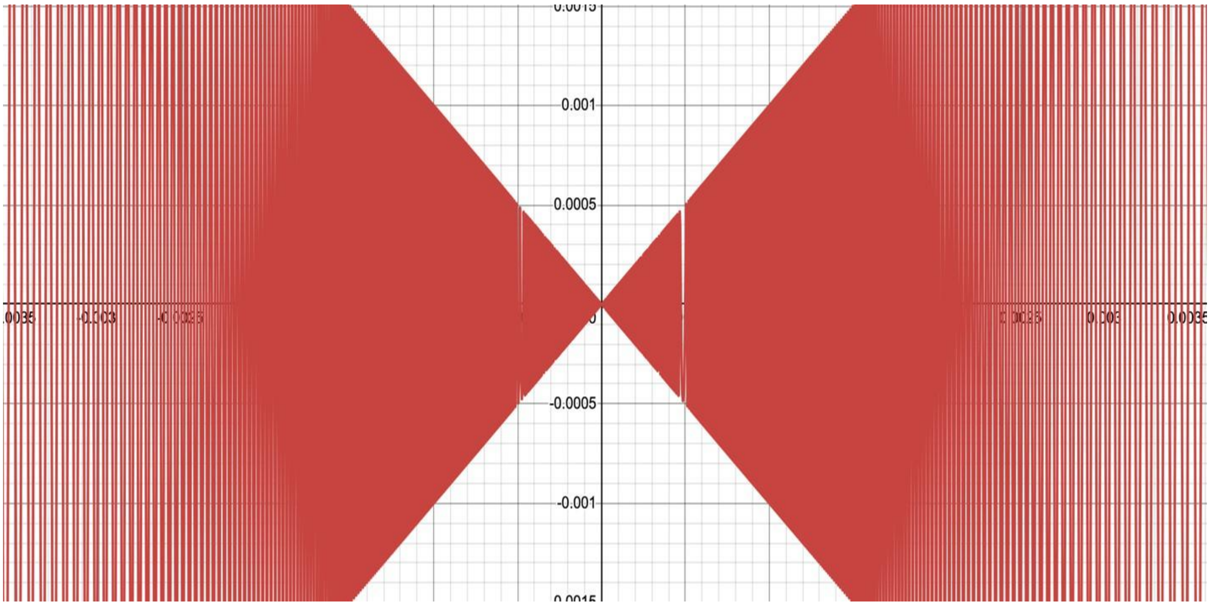
As the function x tends to 0 from the positive side the we can say that $1/x$ approaches to 0 which means that sin$1/x$ will oscillate between -1 and 1 infinitely often [7]. We do not have to worry about the factor x as when the limit tends to 0 the factor also tend to 0 . A nice way to write this is with the help of bounding the function like-
$-x \leq x \sin 1/x \leq x$
Through this question I really wanted to bring forth the importance of a theorem that I have not talked about in the previous examples. The Squeeze Theorem says that if a function is squeezed between two functions that approach the same limit, the function itself must approach that limit. Due to this reason the limit from 0 positive tends to 0 [7]. I really like this question because of the graph of the function if we see the image of the graph on the next page .
This question is I feel a great demonstration of the fact that how close the function comes the value it is approaching but never equals [20]. The unique thing about an oscillatory function like this is that the constant fluctuation of the function make it quite difficult for us to assign a specific value . By this observation we can say that even though the function is itself bounded it never settle on a value. The other thing that makes oscillatory functions unique is how the frequency of the function changes which makes it difficult for us to predict the long term behaviour of the function.



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Now lets analyse the function as x tends to infinity from the negative side . As from the graph you can guess that the function is quiet identical from the left and the right side and the function follows an identical path when it is approaching from the negative side. So as x tends to negative infinity the function behaves in the same way . So when x tends to negative infinity, $1/x$ tends to zero, and similarly, sin $(1/x)$ continues to oscillate between -1 and
1. Because of this, the function x sin $(1/x)$ remains bounded by -x and x, leading to the conclusion by the Squeeze theorem that:
lim x → -∞ x sin $1/x \;= 0$
This symmetry highlights a crucial property of oscillatory functions: their limiting behaviour can be determined despite their continuous fluctuations. The periodicity of the sine function combined with the decay of the x factor ensures that the oscillations become less pronounced as x moves further from zero[3][5]. However, unlike polynomial or exponential functions, oscillatory functions do not settle into predictable growth pattern, making them valuable in mathematical analysis, physics, and signal processing [20][21].

Understanding this function is particularly useful when studying limits because it demonstrates how a function can approach a value without ever stabilizing at it [4][6][7]. This idea is seen in wave phenomena, alternating current signals, and even in quantum mechanics, where oscillatory behaviour plays a fundamental role in describing physical systems [20]. The function x sin(1/x) offers a concrete example of how infinity interacts with periodicity, reinforcing the importance of asymptotic analysis in grasping complex mathematical behaviours [8][25].

Example Question {original}

Below are question that would help a student learn about how limits work institutively .

Question 1
Imagine a scientist is studying the spread of a new virus in a population. The number of infected individuals at time
t (in days) follows the function:

$P(t) = P_0 e^{kt}$
where P_0 is the initial number of infected individuals and k is a positive constant representing the rate of spread.

If no external factors (such as vaccines or social distancing) limit the spread, what do you think happens to $P(t)$ as t
$\rightarrow \infty$?
Write the mathematical expression for the limit, but do not solve it:
$lim_{t\rightarrow\infty}\, P(t) =$

The answer is ∞.

This question helps students understand exponential growth and its implications in real-world scenarios, such as virus outbreaks. It reinforces the concept that certain functions grow unbounded and never level off, a key idea in asymptotic behaviour. Additionally, by asking students to think about the limit without solving, the question encourages conceptual reasoning over mechanical computation. Understanding exponential growth is crucial for applications in epidemiology, finance, and population dynamics [1][3][6][8].

Question 2

A deep-sea exploration team is studying the intensity of light as it penetrates ocean water. The intensity of light $I(d)$
at a depth d meters below the surface follows an inverse proportionality rule:

$$I(d) = \frac{I_0}{1 + \mathrm{kd}}$$

where I_0 is the intensity of light at the surface, and k is a positive constant representing the rate of light absorption by
water.

If the exploration team continues to descend indefinitely, what happens to the light intensity? What do you think is the mathematical limit of $I(d)$ as d→ ∞ ?
Write only the mathematical expression for the limit, but do not solve it:

$lim_{d\to\infty}\ I(d) =$
The answer is 0.

This question helps students develop a deeper understanding of inverse proportionality and its connection to limits in a real-world scenario[1][2][3]. By analysing how light intensity decreases with depth, students can see how inverse relationships model situations where a quantity diminishes indefinitely but never reaches a strict stopping point. The question also reinforces the importance of asymptotic behaviour, showing that as depth increases indefinitely, the intensity of light approaches zero, making it a valuable application in physics, oceanography, and other scientific fields [25][26].

Question 3

A tech company is analysing the efficiency of its data compression algorithm. The amount of space saved, $S(n)$,when compressing a file with n kilobytes of data follows the function:

$S(n) = \mathrm{C}\log(\mathrm{n} + 1)$
where C is a positive constant that depends on the compression method.
As the size of the file increases indefinitely, the space saved also increases, but at a decreasing rate.

What do you think happens to $S(n)$ as n → ∞ ? Write the mathematical expression for the limit but do not solve it:
$lim_{n\to\infty}\ S(n) =$
The answer is ∞.
Since the logarithm function grows indefinitely, albeit at a decreasing rate, the space saved by compression continues to increase without bound as n becomes very large.

This question helps student understand the concept of logarithmic growth in a practical setting.

Unlike linear or exponential growth, logarithmic growth increases without bound but at a slower rate [8][23]. By analysing this compression, students can see how logarithmic functions model real-world processes where growth slows over time, such as data storage, population studies, and information theory [20][21]. This reinforces the importance of asymptotic behaviour in understanding the long-term effects of mathematical models [25][26].

Question 4

A physicist is studying the motion of a pendulum under ideal conditions. The displacement of the pendulum from its equilibrium position over time is modelled by the function:

$f(t) = \mathrm{A}\cos(\omega \mathrm{t})$

where :
A is the maximum displacement (amplitude),
ω is the angular frequency,
t represents time in seconds.
The physicist wants to understand the long-term behaviour of the pendulum's displacement. What happens to f(t) as
$t \to \infty$?
Write the mathematical expression for the limit, but do not solve it:

$lim_{t\to\infty} f(t) =$
This question encourages students to think about oscillatory behaviour and how some functions do not settle to a single value as time progresses [1][2][8]. Unlike exponential or polynomial functions, oscillatory functions like cosine do not approach infinity or zero but continue fluctuating indefinitely [20][25]. This introduces the idea that not all functions have a conventional limit, which is essential for understanding periodic motion, waves, and signal processing in physics and engineering [21][26].

Question 5

A researcher is analysing how different types of growth affect technological advancements over time. Consider the following three functions, where t represents time in years:
1. The number of global internet users follows:
$U(t) = U_0 e^{kt}$
where U_0 is the initial number of users, and k is a positive constant.

2. The efficiency of a machine-learning algorithm improves as:
$E(t) = \log(\mathrm{t} + 1)$
3. The time required to complete a computing task decreases as:
$T(t) = \dfrac{\mathrm{C}}{\mathrm{t}+1}$
where C is a constant representing the initial time required.
Without solving, rank the three functions in order of their rate of growth as $t \to \infty$, from fastest to slowest. Explain your reasoning in one or two sentences.
The answer is 1 > 2 > 3 from the fastest to the slowest.

This question encourages students to develop an intuitive understanding of how different types of functions grow over time. Exponential growth increases the fastest, logarithmic growth increases but slows down over time, and inverse proportionality decreases towards zero. By comparing these functions, students build intuition for real-world applications such as population growth, algorithmic improvements, and computational efficiency [20][25].

Peer – Led Learning : The impact of Teaching your peers

As a student, I have always been ahead of my peers, and whenever I get the chance, I always tend to teach other students. At UCSB, there is a math lab where students of any math levels can come to solve their doubts. As a person, I never had many doubts, and so after solving my questions, I would often go from table to table just to help other students . During that time, I experienced that teaching other people would actually make me revise my own concepts as well as make me learn the perspectives of others. This, in turn, enhanced my thought process on how to look at a question and how looking at the same question from a different prospective could provide new insight.
Research supports this experience by highlighting the significant benefits of peer teaching. For example, the phenomenon known as the "protégé effect" demonstrates that teaching improves the teacher's own learning and understand of material (Effectiviology) [30]. Additionally, studies show that peer teaching allows students to take ownership of their learning which make then develop new ways to tackle problems (Together Plantform) [33].

Another study shows that learning by teaching others is effective as it requires retrieval and organization of prior knowledge. This enhances understanding and further cements it in the teachers mind (Impact Image Marketing)[28]

. As a result in my own experience I have seen that helping others reinforced my grasp of concepts. Peer teaching in classes should be highly encouraged as it also improves communication, providing students with diverse perspectives and techniques for understanding material (WGU) [31].

In my own little experience I have seen how peer studying and teaching other people can help you and boost your confidence. Even for the students who think they cannot teach other as they do not know the content will still be able to learn if they try to make their peers understand something. Personally when I was studying differentiation in class I taught my peers . There is a very important difference between teaching your peer and revising the same concept on your own as when you are teaching someone you have a bit of pressure on you because you do not want to be wrong. This pressure should not be seen negatively because this is the same pressure I you will feel during an exam. Practising under pressure significantly helped me in exams and also [30][28].

The part that truly inspired me was teaching students that necessarily did not do well in the class. I found out that these students had a way of thinking that even I did not have. I loved to see the different approaches each of my peers took to solve a question and how their idea of the question was way different than my [31][33].

This experience taught me that while teaching the topic of asymptotic limits is a very delicate topic and it might interest a particular student or not depends on how it is taught. Students need to be told that asymptotic behaviour of limits is important as Differentiation deals with rates of change which relies heavily on the concept of limits [1][2]. Furthermore, the idea of approaching a value, rather that reaching it, is directly related to the concept of continuity, a key aspect of differentiation [3][8]. If a function has asymptotic behaviour – whether it approaches a finite value, zero, or infinity - it has implications for its smoothness and differentiability. For example, a function that approaches infinity may not have a derivative in a traditional sense at that point, or a function with a limiting value might be continuous but not differentiable at that value [7][20][26].
All these reason make this topic important and is the foundation of the students. A student who has not studied this concept clearly and understood it will face a lot of problems in the future courses. The concept of limits is like the base of the building and if the base of the building is not good enough then the building will fall eventually [25].

Conclusion

Understanding the asymptotic behaviour of limits before learning differentiation is crucial for students, as it builds a strong conceptual foundation for analysing function behaviour, continuity, and rates of change [1][2][8] and provides the students with the intuitive thinking of the concept [3][4]. By mastering limits, students would be able to gain inadept knowledge of the Calculus and have a strong foundation [1][26]. The concept of infinity forms the foundation of limits and calculus. Mastering it helps students become more comfortable with abstract mathematics that cannot be directly observed or visualized [12][19].

This paper emphasise that how the core concept of Limits can help a student grasp difficult concepts of calculus and provides a way for anyone who wants to teach the topic. All examples and question provided in this paper are aligned for developing an intuitive understanding of limits, where intuitive refers to a natural, instinctive grasp of concepts without requiring formal proof [3][7].

There are several areas related to limits that warrant further study to enhance students' understanding before they delve into differentiation. One important topic is the extension of limits to multivariable calculus, where students must grasp how limits behave in multiple dimensions and their role in partial differentiation [25][26]. Additionally, while this paper emphasizes intuitive learning, a deeper exploration of the rigorous epsilon-delta definition of limits can provide a more formal mathematical foundation . Moreover, comparing intuitive learning approaches to formal proof based methods may shed light on the most effective ways to teach limits [6][34]. Finally, addressing common misconceptions students have when first encountering limits can help educators develop strategies to prevent misconceptions [5][30][31].

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