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On A Method for Calculating Forced Bending Vibrations of Beams

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Abstract
The aim of this work is to develop a methodology for calculating forced bending vibrations of elastic beams with continuously distributed external load along its length and non-uniformly distributed concentrated masses. The result of applying the methodology is mathematical formulas describing the dynamic motion of an elastic beam. At the same time, the possibility appears to control its operation to reduce the stress-strain state. Such structural elements are parts of devices that must ensure reliable and safe operation of the mechanism with the possibility of choosing optimal dimensions and material properties. Flawless performance and efficiency will allow the creation of mechanical engineering units, building spans and floors with local load, aviation products and modules with high-quality resistance to increased vibration impacts and many other devices.
Keywords: *mathematical model, attenuation, bending vibrations, rod, beam, elastic material, bending distributed load, boundary conditions.*

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I. INTRODUCTION

The study of bending vibrations of rods taking into account concentrated loads and internal friction in the material structure is an important task, since this component represents one of the most common engineering modules in the fields of industrial, civil, railway, aviation and bridge construction. The problem of vibration isolation in ship hulls and aircraft due to inertial forces of unbalanced machine parts leads to bending problems of thin elastic elements. The ability to influence the motion of such elements is a necessary factor for designing, ensuring safety and reliability of structures made from them. Assessment of internal friction in the physical environment of the material will allow a more accurate analysis of the dynamic properties of the rod and account for energy losses when calculating its stability and durability. The analysis of bending vibrations of beams taking into account discrete concentrated masses, non-uniformly distributed along its entire axis external periodic loads and internal friction in the material structure is relevant from the point of view of developing new calculation methods during the design of such elements.

1.1 Problem statement

The object of research is the plane bending of a rod, the vibrations of which occur in one of the principal planes of the rod. The vertical plane of bending is the coordinate plane *Oxy*. In this coordinate system, the well-known equation of bending vibrations of a beam is considered

$$EJ\frac{\partial^4 w}{\partial x^4}+\rho F\frac{\partial^2 w}{\partial t^2}+\mu_0 EJ\frac{\partial^5 w}{\partial t\partial x^5}-\rho J\frac{\partial^4 w}{\partial t^2\partial x^2}+\chi_a\frac{\partial w}{\partial t}=\boldsymbol{F}(x,t), \tag{1}$$

obtained on the basis of the modified Hooke's law regarding the dependence of stress on deformation and its rate in the structural material according to the proposed viscous friction hypothesis of Woldemar Voigt [1-4] of the form:

$$\sigma = E(\varepsilon+\mu\frac{\partial\varepsilon}{\partial t})\,. \tag{2}$$

Here E – is the Young's modulus, J – is the static moment of inertia of the beam, ρ – is the density of the beam material, F – is the area of the bending cross-section, μ – is the internal friction coefficient of the material, μ_0 – is its reduced analogue, χ_a – is the external friction coefficient of the material, σ – is the stress, ε – is the value of relative deformation, x – is the coordinate of the beam centerline, t – is the time parameter, $w(x,t)$ – is the deflection of the beam centerline. The external intensive force $\boldsymbol{F}(x,t)$ is continuous along the entire length of the beam and is distributed along it in the form of a pulsating loading regime having a harmonic time dependence

$$\boldsymbol{F}(x,t) = q_1(x)\sin(kt) + q_2(x)\cos(kt) + \sum_{i=1}^{n} f_i(x_i,t)\,. \tag{3}$$

At points x_i on the beam are concentrated discrete mass intensities m_i ($i = \overline{1,n}$), possessing the same inertia as its particles:

$$f_i(x_i,t) = -m_i \frac{\partial^2 w(x_i,t)}{\partial t^2}\,. \tag{4}$$

1.2 Procedure for calculation of bending vibrations of elastic beams

Forced vibrations of a loaded beam with discretely concentrated masses are investigated. Solutions of equation (1) should be considered in accordance with the regularities (3) of the forcing force

$$w(\xi,t) = \varphi(\xi)\sin(kt) + \psi(\xi)\cos(kt)\,. \tag{5}$$

This desired variable $w(x,t)$ is dimensionless with respect to the linear parameter $x \in [0, l]$ with replacement by a new variable $\xi \in [0, 1]$

$$\xi = \frac{x}{l}\,. \tag{6}$$

As a result, the original equation (1) in accordance with formulas (3) and (5) will lead to a system of inhomogeneous fourth-order differential equations [1]:

$$\begin{aligned} &\frac{\partial^4\varphi}{\partial\xi^4} + b\frac{\partial^2\varphi}{\partial\xi^2} - \beta^2\varphi - \mu\frac{\partial^4\psi}{\partial\xi^4} - \mu_a\psi = g_1(\xi), \\ &\frac{\partial^4\psi}{\partial\xi^4} + b\frac{\partial^2\psi}{\partial\xi^2} - \beta^2\psi + \mu\frac{\partial^4\varphi}{\partial\xi^4} + \mu_a\varphi = g_2(\xi), \end{aligned} \tag{7}$$

where $b = \dfrac{l^2k^2\rho}{E}$, $\beta^2 = \dfrac{l^4k^2}{EJ}\rho F$, $\mu = \mu_0 k$, $\mu_a = \dfrac{k\chi_a l^4}{EJ}$, $g_1(\xi) = \dfrac{l^4}{EJ}\left(q_1(\xi) + k^2\sum_{i=1}^{n} m_i\varphi(\xi_i)\right)$, $g_2(\xi) = \dfrac{l^4}{EJ}\left(q_2(\xi) + k^2\sum_{i=1}^{n} m_i\psi(\xi_i)\right)$, $q_1(\xi) = q_{11}\sin(\kappa\xi)$, $q_2(\xi) = q_{21}\cos(\kappa\xi)$.

To solve the system of linear inhomogeneous differential equations (7) in general form, its canonical form is used [5-15]

$$\begin{aligned} &L_1(\varphi) - d_{12}(\psi) = g_1(\xi), \\ &d_{21}(\varphi) + L_2(\psi) = g_2(\xi), \end{aligned} \tag{8}$$

where $L_1(H) = L_2(H) = \dfrac{d^4H}{d\xi^4} + b\dfrac{d^2H}{d\xi^2} - \beta^2\cdot H$, $d_{12}(H) = d_{21}(H) = \mu\dfrac{d^4}{d\xi^4}H + \mu_a\cdot H$ – is a set of differential operators. In these notations, it is quite simple to separate the equations with respect to variables $\varphi(\xi)$ and $\psi(\xi)$ by introducing a new common function

$$D(\varphi(\xi)) = L_1(L_2(\varphi)) + d_{12}(d_{21}(\varphi)) = \left(1+\mu^2\right)\frac{d^8\varphi}{d\xi^8} + 2b\frac{d^6\varphi}{d\xi^6} + (b^2 - 2\beta^2 + 2\mu\mu_a)\frac{d^4\varphi}{d\xi^4} - 2\beta^2 b\frac{d^2\varphi}{d\xi^2} + \mu_a^2 + \beta^4. \tag{9}$$

Then system (8) will become as follows:

$$D(\varphi(\xi))=\frac{l^4}{EJ}\left(\frac{\partial^4\left(g_1+\mu g_2\right)}{\partial\xi^4}+b\frac{\partial^2 g_1}{\partial\xi^2}-\beta^2 g_1+\mu_a g_2\right),$$
$$D(\psi(\xi))=\frac{l^4}{EJ}\left(\frac{\partial^4\left(g_2-\mu g_1\right)}{\partial\xi^4}+b\frac{\partial^2 g_2}{\partial\xi^2}-\beta^2 g_2-\mu_a g_1\right). \qquad (10)$$

Based on formula (9), a general characteristic equation is obtained

$$\left(1+\mu^2\right)\lambda^8+2b\lambda^6+(b^2-2\beta^2+2\mu\mu_a)\lambda^4-2\beta^2 b\lambda^2+\mu_a^2+\beta^4=0\,, \qquad (11)$$

from which eight complex-conjugate roots are calculated

$$\lambda_{1,2}^2=\frac{1}{2}\left(i\mu-b\pm\sqrt{\left(b-i\mu\right)^2+4\left(\beta+i\mu_a\right)}\right),\quad \lambda_{3,4}^2=\frac{1}{2}\left(-i\mu-b\pm\sqrt{\left(b+i\mu\right)^2+4\left(\beta-i\mu_a\right)}\right). \qquad (12)$$

The fundamental system of solutions of differential equations (8) will consist of eight functions $u_j(\xi)$, $j=\overline{1,8}$. The solutions of system (7) in general form will be as follows:

$$\varphi(\xi)=C_1u_1(\xi)+C_2u_2(\xi)+C_3u_3(\xi)+C_4u_4(\xi)+C_5u_5(\xi)+C_6u_6(\xi)+C_7u_7(\xi)+C_8u_8(\xi)+$$
$$+\alpha_1\sin(\kappa\xi)+\alpha_2\cos(\kappa\xi)+\eta_1P+\eta_2Q,$$
$$(13)$$
$$\psi(\xi)=R_1u_1(\xi)+R_2u_2(\xi)+R_3u_3(\xi)+R_4u_4(\xi)+R_5u_5(\xi)+R_6u_6(\xi)+R_7u_7(\xi)+R_8u_8(\xi)+$$
$$+\gamma_1\sin(\kappa\xi)+\gamma_2\cos(\kappa\xi)+\varsigma_1P+\varsigma_2Q,$$

where the quantities $P=\frac{l^4k^2}{EJ}\sum_{i=1}^{n}m_i\varphi(\xi_i)$, and $Q=\frac{l^4k^2}{EJ}\sum_{i=1}^{n}m_i\psi(\xi_i)$ are constants, C_j, R_j, ($j=\overline{1,8}$), η_1, η_2, ζ_1, ζ_2, α_1, α_2, γ_1, γ_2 – are indeterminate parameters of the system of differential equations (7), and the last four α_1, α_2, γ_1, γ_2 are determined by the forced nature of vibrations of the pulsating loading regime.

For complete determination of all 24 unknown parameters, eight zero conditions for natural vibrations, eight equivalence conditions for forced vibrations including constants *P* and *Q*, and eight boundary conditions for the beam should be used. As such, the case of a supported right end at $\xi = 1$ is chosen, which is described by the following boundary conditions [1]:

$$\varphi(1)=0.0,\qquad \psi(1)=0.0,\qquad \partial^2\varphi(1)/\partial\xi^2=0.0,\ \ \partial^2\psi(1)/\partial\xi^2=0.0, \qquad (14)$$

$$\partial\varphi(1)/\partial\xi=\theta_1,\ \ \partial\psi(1)/\partial\xi=\theta_2,\ \ \partial^3\varphi(1)/\partial\xi^3=\theta_3,\ \ \partial^3\psi(1)/\partial\xi^3=\theta_4, \qquad (15)$$

here θ_1, θ_2, θ_3, θ_4 are constant values. In this case, the left end at $\xi = 0$ is considered free. As a result of these remarks, the desired functions $\varphi(\xi)$ and $\psi(\xi)$ will become linearly dependent only on the coordinates of the application points of concentrated masses mi , where m_i ($i=\overline{1,n}$):

$$\varphi(\xi)=f_1(\xi,\quad m_1g_1+m_2g_2+m_3g_3+...+m_ng_n,\quad m_1g_{n+1}+m_2g_{n+2}+m_3g_{n+3}+...+m_ng_{2n}),$$
$$\psi(\xi)=f_2(\xi,\quad m_1g_1+m_2g_2+m_3g_3+...+m_ng_n,\quad m_1g_{n+1}+m_2g_{n+2}+m_3g_{n+3}+...+m_ng_{2n}), \qquad (16)$$

where $g_i = \varphi(\xi_i)$, $g_{n+i} = \psi(\xi_i)$ ($i=\overline{1,n}$). Composing a system of linear algebraic equations in the amount of $2n$ based on formulas (16), $2n$ unknown variables g_j are found mutually unambiguously for $j=\overline{1,2n}$

$$g_i=f_1(\xi_i,\quad m_1g_1+m_2g_2+m_3g_3+...+m_ng_n,\quad m_1g_{n+1}+m_2g_{n+2}+m_3g_{n+3}+...+m_ng_{2n}),$$
$$g_{n+i}=f_2(\xi_i,\quad m_1g_1+m_2g_2+m_3g_3+...+m_ng_n,\quad m_1g_{n+1}+m_2g_{n+2}+m_3g_{n+3}+...+m_ng_{2n}). \qquad (17)$$

The found values when substituted into formulas (13) and (5) are the solution $w(\xi, t)$ of the posed problem (1).

II. RESULT AND DISCUSSION

A test example of implementing the presented calculation methodology.

The following parameters are used to verify the calculation and visually observe the displacements of beam points: $E = 2.1 \cdot 10^7$ N/m^2, $\rho = 7800.0$ kg/m^3, $r = 0.02$ m, $l = 1.8$ m, $F = 0.001256637062$ m^2, $k = 15.2$ rad./s, $\kappa = 0.5$ rad., $J_i = 1.256637062 \cdot 10^{-7}$ m^4, $b = 0.2780401371$, $\beta = 94.91312051$, $\mu = 579120.0$ kg/(m s), $\mu_0 = 38100.0$ kg/m, $\mu_a = 605.3761266$, $\chi_a = 10.012$ kg/(m s), $n = 4$, $m_1 = 3$ kg/m, $m_2 = 2$ kg/m, $m_3 = 5$ kg/m, $m_4 = 4$ kg/m.

As boundary conditions, the case of a supported right end at $\xi = 1$ with the following characteristics was chosen:

$$\varphi(1) = 0.0, \quad \partial\varphi(1)/\partial\xi = 0.07, \quad \partial^2\varphi(1)/\partial\xi^2 = 0.0, \quad \partial^3\varphi(1)/\partial\xi^3 = -0.3, \tag{18}$$

$$\psi(1) = 0.0, \quad \partial\psi(1)/\partial\xi = -0.04, \quad \partial^2\psi(1)/\partial\xi^2 = 0.0, \quad \partial^3\psi(1)/\partial\xi^3 = 0.5. \tag{19}$$

In this case, the left end $\xi = 0$ is considered free.

The external force intensity $\boldsymbol{F}$ in coordinates (ξ, t) is represented by a pulsating load of the following form:

$$\boldsymbol{F}(\xi,t) = 0.05 \cdot \sin(0.5\xi)\sin(15.2t) + 0.07 \cdot \cos(0.5\xi)\cos(15.2t) + \sum_{i=1}^{n} f_i(x_i,t) \cdot \tag{20}$$

At the initial moment of time, it exerts a stationary effect on the elastic beam supported on the right, causing its initial bending (fig.1)

$$\boldsymbol{F}_*(\xi,0) = 0.07 \cdot \cos(0.5\xi) + \sum_{i=1}^{n} f_i(x_i,0) \cdot \tag{21}$$

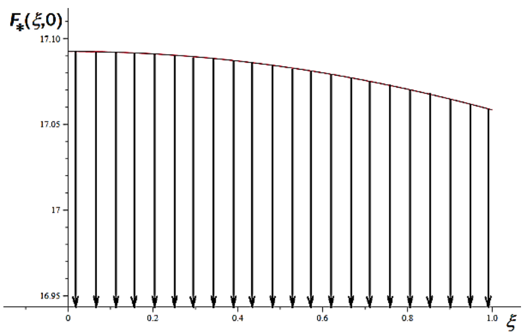

Figure1: External load $\boldsymbol{F}_*(\xi,0)$ continuously distributed along the entire length of the beam with discrete inertial forces of concentrated masses

According to the presented methodology (5)-(17), a general solution to the problem of bending vibrations of an elastic beam carrying four concentrated masses was obtained. The beam is supported on the right side and has a free boundary on the left side

$$w(\xi,t) = \varphi(\xi)\sin(15.2t) + \psi(\xi)\cos(15.2t)\,, \tag{22}$$

where

$$\begin{aligned} \varphi(\xi) = {} & -0.7925360246 \cdot e^{0.3241464412\xi} \cdot \sin(0.1406810746\,\xi) - 2.218314192 \cdot e^{0.3241464412\xi} \cdot \cos(0.1406810746\,\xi) - \\ & -0.2848701563 \cdot e^{-0.3241464412\xi} \cdot \sin(0.1406810746\,\xi) + 4.473818035 \cdot e^{-0.3241464412\xi} \cdot \cos(0.1406810746\,\xi) + \\ & +2.889158400 \cdot e^{0.1406816978\,\xi} \cdot \sin(0.3241467117\,\xi) - 0.3883023529 \cdot e^{0.1406816978\,\xi} \cdot \cos(.3241467117\,\xi) + \\ & +3.358716004 \cdot e^{-0.1406816978\,\xi} \cdot \sin(0.3241467117\,\xi) - 1.920393007 \cdot e^{-0.1406816978\,\xi} \cdot \cos(0.3241467117\,\xi) - \\ & -1.248258252 \cdot 10^{-6} \cdot \sin(0.5\,\xi) + 7.138909833 \cdot 10^{-6} \cdot \cos(0.5\,\xi) + 0.03319339703, \end{aligned} \tag{23}$$

$$\begin{aligned} \psi(\xi) = {} & 2.218314189 \cdot e^{0.3241464412\xi} \cdot \sin(0.1406810746\,\xi) - 0.7925360256 \cdot e^{0.3241464412\xi} \cdot \cos(0.1406810746\,\xi) + \\ & +4.473818030 \cdot e^{-0.3241464412\xi} \cdot \sin(0.1406810746\,\xi) + 0.2848701567 \cdot e^{-0.3241464412\xi} \cdot \cos(0.1406810746\,\xi) - \\ & -0.3883023524 \cdot e^{0.1406816978\,\xi} \cdot \sin(0.3241467117\,\xi) - 2.889158401 \cdot e^{0.1406816978\,\xi} \cdot \cos(0.3241467117\,\xi) + \\ & +1.920393007 \cdot e^{-0.1406816978\,\xi} \cdot \sin(0.3241467117\,\xi) + 3.358716005 \cdot e^{-0.1406816978\,\xi} \cdot \cos(0.3241467117\,\xi) - \end{aligned} \tag{24}$$

$-5.099221311 \cdot 10^{-6} \cdot \sin(0.5\ \xi) - 1.747561553 \cdot 10^{-6} \cdot \cos(0.5\ \xi) - 0.005194136132.$

The concentrated loads are located at the localization points of dimensionless coordinates $\xi_1 = 0.25$, $\xi_2 = 0.4$, $\xi_3 = 0.6$, $\xi_4 = 0.85$ or in coordinates $x_1 = 0.45$ m, $x_2 = 0.72$ m, $x_3 = 1.08$ m, $x_4 = 1.53$ m, presented in dimensional form.

Figures 2 and 3 show the bending vibrations of the beam centerline with concentrated masses at different time moments.

The validity of the obtained formulas can be verified by substituting the found solutions into the original equation for bending vibrations of an elastic beam (1), taking into account the features of internal friction in the material structure.

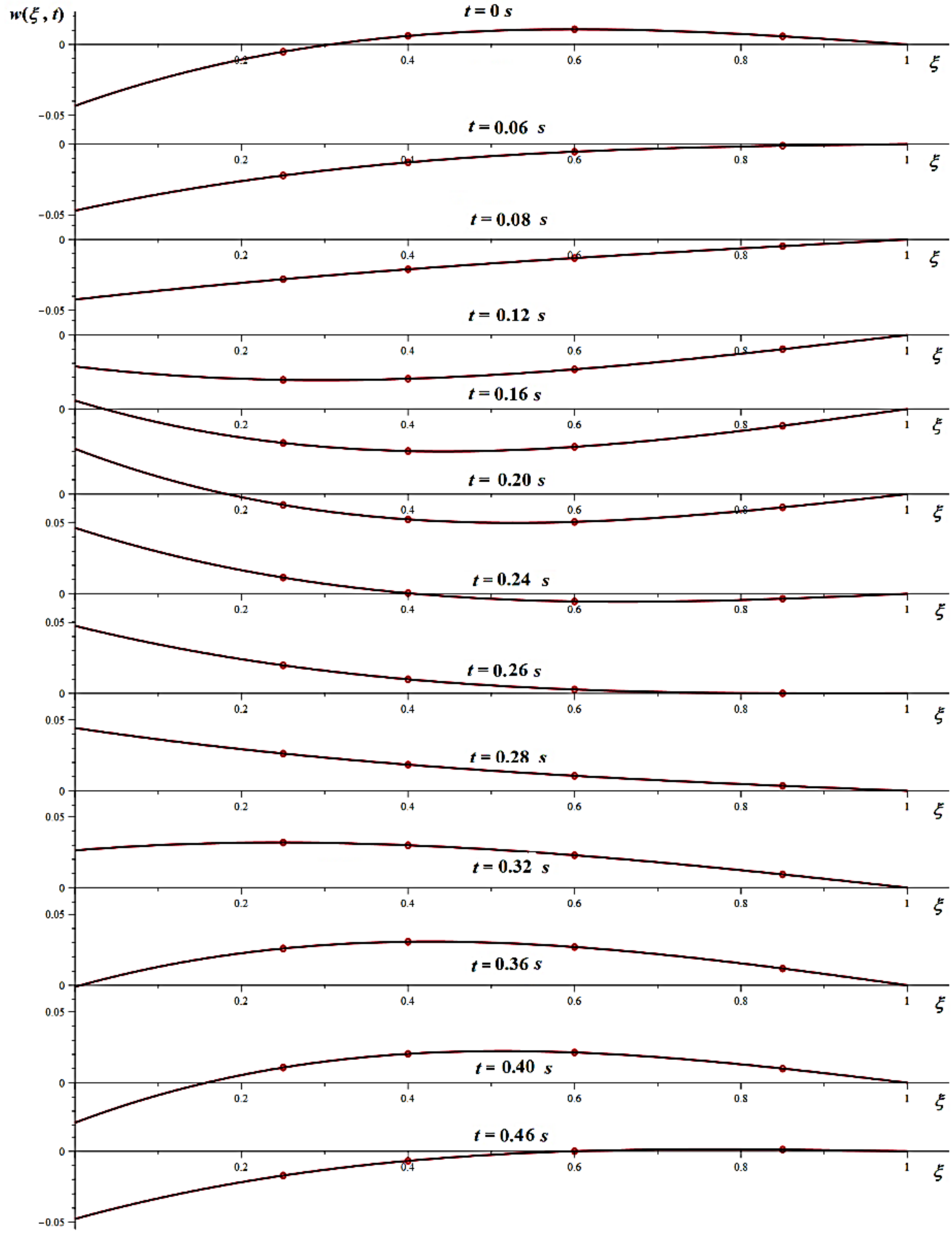

Figure 2: Bending vibrations of the beam centerline with concentrated masses at different time moments

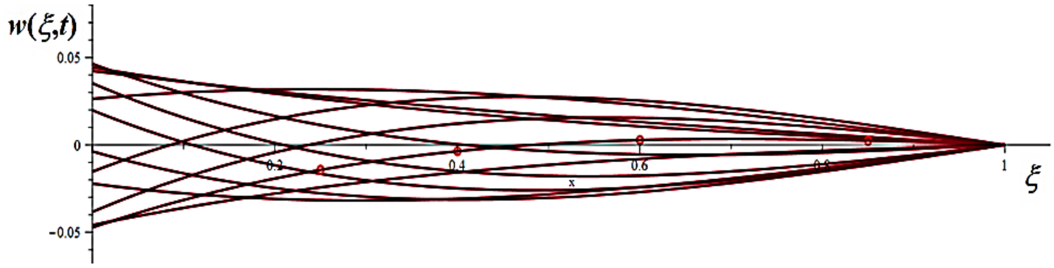

Figure 3: Bending vibrations of the beam centerline with concentrated masses at different time moments in one coordinate system

III. CONCLUSION

A solution to the equation of bending vibrations of an elastic beam with concentrated masses under the action of external forces and internal friction forces with a supported right edge has been found. Graphical illustrations with dynamic graphical animation were obtained for various physical and geometric parameters of mechanical objects, which are rods made of elastic materials. A methodology for calculating forced bending vibrations of elastic beams with continuously distributed external load along its length and discretely distributed concentrated masses has been developed.

From the analysis of the conducted numerical-analytical tests, it follows that external loads and concentrated masses have a significant influence on the bending vibrations of the beam. To predict the behavior of structures under the action of external loads, it is necessary to take into account the elastic properties of the rod material based on the modified Hooke's law. This leads to asymmetric forms of natural and forced vibrations of the rod due to the presence of internal resistance in the material structure and depending on the nature of the applied external load and inertial forces of non-uniformly distributed concentrated loads. Along with this, the frequencies of the bending forms of the beam centerline change, which become explicitly dependent on the viscous friction coefficients of the elastic deformation model proposed by W. Voigt. Such facts are the most reliable and plausible manifestations of the physical properties of solid bodies of tested samples compared to calculations leading to symmetric undamped harmonic vibrations.

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