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Discrete time Markov renewal process

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Abstract
A Markov Renewal Process is defined as a generalization of a renewal process, where the sequence of holding times is not independent and identically distributed. The distributions of holding times depends on the states in a Markov Chain. It is used in various applications such as queuing systems and machine repair problems.
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I.INTRODUCTION

In probability theory and related fields, a stochastic or random process is a mathematical object usually defined as a collection of random variables. Historically, the random variables were associated with or indexed by a set of numbers, usually viewed as points in time etc. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. **[1]**

Renewal theory has assumed great importance because of its theoretical structure as well as for its application in diverse areas. Renewal theory and renewal theoretic arguments have been advanced in a variety of situations such as demography, man power studies, reliability and so on. It is the generalization of Poisson process. A lot of research has been reported in the area of characterizing the superposition process of Renewal and Markovian process. However, to the best of our knowledge characterizing the superposition of Markov Renewal Process has not as yet been addressed. The structure of the interval process resulting from superposing two independent MRP was characterised. The resulting stochastic process has a very large number of states which limits the applicability. **[2]**

The project presents innovative results on the behaviour of a multiplexer with infinite waiting room shared by multiple on/off sources. For the case of homogenous sources, the model can be used for all levels of multiplexer utilization. In the heterogeneous case, the asymptotic decay parameter is a simple function of the first two moment of the on and off periods and peak rates. However, our results show that for the case of multiplexer with finite buffer, more parameters are needed than just the first two moments of the on and off period lengths. **[3]**

This is a detailed study of Markov Renewal Process in discrete time. The leading section involves some basic concepts and results with some examples. As, we move onto the next section, it deals with the study of Markov Renewal Equation, Renewal Equation, Renewal Function, and Interval Transition Probability Matrix.**[4]**

II.PRELIMINARIES

2.1 STOCHASTIC PROCESS

The project dealing with is examples and types of Stochastic process and its superposition. Since last century, there have been marked changes in the approach to scientific enquiries. There has been greater realization that probability models are more realistic than deterministic models in many situations. The scope of application of random variables which are functions of time or space or both has been on the increase.

Family of random variables which are functions of say, time, are known as stochastic process (or random processes or random functions). Example: Consider there are 'r' cells and an infinitely large number of identical balls and that balls are thrown at random, one by one, into the cells, the balls thrown being equally likely to go into any one of the cells. Suppose that 'X_n' is the number of occupied cells after 'n' rows. Then,$\{X_n, n \geq 1\}$constitutes a stochastic process.

2.2 INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM SEQUENCE

Let 'X_n' be an 'i^{th}' random sequence. All 'X_i' has the same distribution. Therefore,$P_{X_i}(\boldsymbol{x}) = P_{\boldsymbol{X}}(\boldsymbol{x})$. For a discrete value process, the sample vector $X_{n_1} \dots X_{n_k}$has the joint p. m. f. (by property of independent random variables).

$$P_{(x(t_1),\dots x(t_n))}(x_1, \dots x_k) = P_x(x_1)P_x(x_2) \dots P_x(x_n) = P_x(x_i), \forall\, i = 1, 2..$$

2.3 TRANSITION PROBABILITIES

Transitional probabilities are conditional probabilities describing the Probability behaviour associated with the transition of the system.

By an m-step transition probability we mean a condition probability of the form,$P\ [X_{n+m} = j | X_n = i]$ and is denoted by $P_{ij}(m)$.

$$P_{ij}(n) = P\ [X_n = j | X_0 = i]$$

$= P\ [X_{n+1} = j | X_1 = i]$ and so on.

$$P_{ij}(1) = P\ [X_{n+1} = j | X_n = i]$$
$$= P[X_1 = j | X_0 = i]$$

and so on is called one step transition probability.

2.4 TRANSITION MATRIX/TRANSITION PROBABILITY MATRIX

The transition probabilities p_{jk} satisfy

$$p_{jk} \geq 0, \sum_k p_{jk} = 1\ \forall\, j.$$

These probabilities may written in the matrix form

$$p = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

This is called the transition probability matrix or matrix of transition probabilities (t. s. p.) of the Markov chain. P is the stochastic matrix i.e., a square matrix with non-negative elements and unit row sums.

2.5 POISSON PROCESS:

Poisson process is a type of discrete valued stochastic process with independent stationary increments. This process has convenient mathematical properties. The Poisson process is often defined on the real line, where it can be considered as a stochastic process. It has a property that each point is stochastically independent to all the other points in process, which is so called purely or completely random process.

2.6 PERIODIC:

The renewal event E* is said to be periodic if there exists an integer m >1 such that E* can occur only at trails numbered m, 2m …. The greatest m with this property is said to be period of E*. It is said to be a periodic if no such m exist.

2.7 LAPLACE TRANSFORM:

Let $f(t)$ be a function of a positive real variable t. Then the Laplace transform(L.T.) of $f(t)$ is defined by

$$\bar{f}(s) = \int_0^\infty \exp(-st)\,.\,f(t)dt$$

for the range of values of s for which the integral exists.

Laplace transform is a generalisation of generating functions.

2.8 MARKOV PROCESS:

If $\{X(t), t \in T\}$ is stochastic process such that, given the value of $X(s)$, the values of $X(t), t > s$ do not depend on the values of $X(u), u < s$, then the process is said to be a Markov process.

2.8.1 Probability Generating Function: Mean and Variance

Suppose that X is a random variable that assumes non-negative integral values 0, 1, 2… and that

$Pr\{X = k\} = p_k, k = 0, 1, 2 \dots, \sum_k p_k = 1$ (1)

If we take a_k to be the probability p_k , $k = 0, 1, 2$ ... then the corresponding probability generating function P(s) = $\sum p_k s^k$ of the sequence of probabilities $\{p_k\}$ is known as probability generating function of the random variable X. It is sometimes also called the s-transform (or geometric transform) of the random variable X.

We have P(1) = 1; the series P(s) converges absolutely and uniformly in the interval $-1 \le s \le 1$ and is infinitely differentiable. The function P(s) is defined by $\{p_k\}$ and in turn defines $\{p_k\}$uniquely, i.e., a p. g. f. determines a distribution uniquely. Again,

$P(s) = \sum_{k=0}^{\infty} \Pr\{X = k\}\, s^k = E(s^x)$ (2)

where E(s^x) is the expectation of the function s^x of the random variable X.

The first two derivatives of P(s) are given by

$P'(s) = \sum_{k=1}^{\infty} kp_k s^{k-1}, -1 < s < 1$ (a)

$P''(s) = \sum_{k=1}^{\infty} k(k-1)p_k s^{k-2}$ (b)

The expectation $E(X)$ is given by

$E(X) = \sum_{k=1}^{\infty} kp_k = \lim_{s\to 1} P'(s) = P'(1)$ (3)

We have,

$E\{X(X-1)\} = \sum_k k(k-1)P_k = \lim_{s\to 1} P''(s)$ (4)

And,

$$E(X^2) = E\{X(X-1)\} + E(X) = P''(1) + P'(1)$$

Hence

$$var\,(X) = E(X^2) - [E(X)]^2$$
$$= P''(1) + P'(1) - [P'(1)]^2 \quad (5)$$

If as $s \to 1, \sum kp_k$ diverges then we say that $E(X) = P'(1) = \infty$.

The relation (3) gives the mean and (5) gives the variance of X in terms of thep. g. f of X. In fact, moments, cumulants etc. can be expressed in terms of generating functions. More generally, the kth factorial moments of X is given by

$$E\{X(X-1)\dots(X-k+1)\} = \frac{d^k}{ds^k}P(s)\,\Big|\; s = 1 for\ k = 1, 2, 3 \dots$$

Note that $P\,(e^t)$ is the moment generating function and P(1+t) is the factorial moment generating function. Further, the p. g. f.$P(s) = E\,(s^x)$ is a special case of the characteristic function $E\,(e^{itx})$.

EXAMPLES:

- ❖ Binomial Distribution: Let X be a random variable denoting the number of success (or failures) in a fixed number n of Bernoulli trials. Then X has a binomial distribution having p. m .f.

$$p_k = \Pr(X = k) = \binom{n}{k} p^k q^{n-k}, k = 0, 1, 2 \dots n$$

The p. g. f. of X is

$$P(s) = (q + sp)^n$$

with $E(X) = np$ and $var\,(X) = npq$.

The binomial random variable has parameters n and p and is the sum of n independent Bernoulli's random variables each with parameter p.

- ❖ Poisson Distribution: Let X be Poisson variates with p. m. f

$$p_k = \Pr\{X = k\} = \frac{\exp(-\lambda)\,.\,\lambda^k s^k}{k!}$$
$$= \exp(-\lambda).\exp(\lambda s)$$
$$= \exp\{\lambda(s-1)\}$$

- ❖ Gamma Distribution: Let X have a two parameter gamma distribution with parametersλ, k then density function of random variable X is

$$f_{\lambda,k}(x) = \frac{\lambda^k x^{k-1}\exp(-\lambda x)}{\Gamma(x)}, x > 0$$
$$= 0 \qquad , x < 0$$

We shall consider here kind of generalisation of a Markov process, as well as of renewal process. We first consider a Markov process with discrete state space and its generalisation by a way of random transformation of the time scale of process. Let $\{X\,(t)\}$ be a Markov process with discrete state space and let the transitions occur at epochs (or instants of time) $t_i, i = 0, 1, 2$. Thenthe sequence $\{X_n = X\,(t_n+)\}$ forms a Markov chain and the time intervals$(t_{(n+1)} - t_n) = T_n$between transitions have independent exponential distributions, the parameters of which may depend on X_n.

Let the states of the process be denoted by the set $E = \{0, 1, 2,\}$ and let the transitions of the process occurs at epochs (or instants of times)$t_0 = 0, t_1, t_2$,.. Let X_n denote the transition occurring at epoch t_n. If

$$P\{X_{n+1} = k, t_{n+1} - t_n \le t \mid X_0 = \cdots, X_n = \cdots, t_0, \cdots, t_n\}$$

$$P\{X_{n+1}=k, t_{n+1}-t_n \le t \mid X_n = \cdots..\},$$

then$\{X_n, t_n\}, n=0,1,2,....$ is said to constitute a Markov renewal process with state space E. $\{X_n, T_n\}$ is also used to denote the process $\{X_n, t_n\}$. Assume that the process is temporally (or time) homogeneous, i.e.
$P\{X_{n+1}=k, t_{n+1}-t_n \le t \mid X_n = j\} = Q_{jk}(t)$is independent of n.

Define,

$$P_{jk} = \lim_{t\to\infty} Q_{jk}(t) = P\{X_{n+1} = k \mid X_n = j\}$$

The matrix $Q(t) = (Q_{jk}(t))$ is known as the semi-Markov Kernal of Markov renewal process$\{X_n, t_n\}$.
Proposition:
$\{X_n, n = 0,1 ...\}$constitutes a Markov chain with state space E and transition probability matrix $P = (p_{jk})$.

The continuous parameter $\{Y(t)\}$ with state space E, defined by

$$Y(t) = X_n, t_n \le t \le t_{n+1}$$

is called semi Markov process. The Markov chain $\{X_n\}$ is said to be an embedded Markov Chain of the semi Markov process. $\{X_n\}$refers to the state of the process at transition occurring at epoch t_n and $Y(t)$ that of the process at its most recent transition. Consider, a simple random walk performed by a particle. Suppose that $t_0, t_1, ...$ are the epochs when transitions take place and that X_n is the transition at the epocht_n; also that at $t_n$the particle makes a move to the right (and enters the state j), at $t_{n+1}$to the left (enters the state j-1), at $t_{n+2}$to the right (enters the state j) and so on. Then,

$Y(t) = j \qquad t_n \le t < t_{n+1}$
$= j-1 \qquad t_{n+1} \le t \le t_{n+2}$
$= j \qquad t_{n+2} \le t \le t_{n+3}$
$= j+1 \qquad t_{n+3} \le t \le t_{n+4}$
$= j+2 \qquad t_{n+4} \le t \le t_{n+5}$and so on.

2.9 RENEWAL FUNCTION.
The mean function of the renewal process is defined as the renewal function and is denoted by $M(t)$.

$$M(t) = E(N(t))$$

2.9.1 Renewal equation
The renewal function M(t) satisfies the equation,

$$\mathrm{M(t)} = \mathrm{F(t)} + \int_0^t M(t-x)dF(x)$$

where $F_n(t) = P[S_n \le t]$.
Proof:
Let time duration for first renewal denoted by $X_1 = x$, we get

$$M(t) = E(N(t)) = \int_0^\infty E[N(t) \mid X1 = x] d\,F(n)$$

Now suppose that $> t$, means first renewal time is greater than t.
So that $[E\{N(t)\} \mid X_1 = x] = 0$
Now suppose that $\le t$, then first renewal has been taken place by the time X and during the remaining $t-x$;
Expected number of renewal will be $M(t-x)$
Therefore,

$$E[N(t)|X_1 = x] = 1 + M(t-x)$$

$$M(t) = \int_0^t E[N(t)|X_1 = x]\,d\,F(x) + \int_t^\infty E[N(t)|X = x]\,d\,F(x)$$

$$= \int_0^t (1 + M(t-x))\,d\,F(x) + 0$$

$$= \int_0^t d\,F(x) + \int_0^t M(t-x)\,d\,F(x)$$

$$= [(x)]_0^t + \int_0^t M(t-x)\,d\,F(x)$$

$$M(t) = F(t) + \int_0^t M(t-x)\, d\, F(x) [since\ F(0) = 0]$$

2.9.2 Generalised form of renewal equation

An integral equation of the form $\upsilon(t) = g(t) + \int_0^t \upsilon(t-x)\, dF(x)$ is general form of the renewal equation. Here the function $g(t)$ and the function *F(x)* are known and $\upsilon(t)$ be the unknown quantity.

2.10 MARKOV RENEWAL EQUATION

For all i,j for $t \geq 0$

$$f_{ij}(t) = \delta_{ij} h_i(t) + \sum_k \int_0^t f_{kj}(t-x) dQ_{ik}(x)$$

where$h_i(t) = 1 - \sum_k Q_{ik}(t)$

$$= 1 - W_i(t)$$
$$= P\{Ti > t\}$$

Proof:

By conditioning on time $t_1$on first equation,

$f_{ij}(t) = P\{Y(t) = j \mid X(0) = i\}$

$$= \sum_k \int_0^\infty P\{Y(t) = j \mid X_0 = i, X_1 = k, t_1 = x\} dQ_{ik}(x)$$

If $t_1 = x > t$, i.e., the first transition epoch is beyond, then $Y(t) = Y(0) = i$

If $t_1 = x < t$,then starting from state k, the process ends up with state j by the remaining time $(t-x)$. Thus,

$$P\{Y(t) = j \mid X_0 = i,\ X_1 = k, t_1 = x\}$$
$$= \delta_{ij}\ , if\ x > t$$
$$= f_{kj}(t-x)\ , if\ x \leq t$$

Hence,

$f_{ij}(t) = \sum_k [\int_0^t f_{ij}(t-x) dQ_{ij}(x) + \delta_{ij} \int_t^\infty d\, Q_{ik}(x)]$ (2)

Since $\sum_k \int_0^\infty d\, Q_{ik}(x) = 1$, the second term in (2) becomes

$$\delta_{ij}\{1 - \sum_k \int_0^t d\, Q_{ik}(x)\} = \delta_{ij}\{1 - \sum_k Q_{ik}(t)\}$$
$$= \delta_{ij} h_i(t)$$

and thus (1) in the theorem is obtained. Hence Markov renewal equation.

2.11 INTERVAL TRANSITION PROBABILITY MATRIX

The matrix with functions of $t, k_{ij}(t),\ i,j = 1,2, \ldots .m$ as elements will be denoted by $K(t)$, ie, $K(t) = (k_{ij}(t))$. The matrices obtained by taking derivatives, integrals, Laplace transforms etc. of the elements will be represented accordingly. For example,

$$\int_0^t K(x)dx = \int_0^t K_{ij}(x)dx$$
$$K^*(s) = K_{ij}^*(s),$$

where*K*(.)*denote Laplace transform.

We shall refer to these matrices here by

$$F(t) = (f_{ij}(t))$$
$$C(t) = (C_{ij}(t))$$

where$C_{ij}(t)dt = P_{ij}(t)dk_{ij}(t) = dQ_{ij}(t)$

$A(t) = (a_{ij}(t))$ where$a_{ij}(t) = \delta_{ij} h_i(t)$ (1)

The matrix A is a diagonal matrix.

Since $f_{ij}(t) \geq 0 \& \sum_j f_i(t) = 1$ for all i =1, 2…., m the matrix $F(t)$ is a stochastic matrix, it is called inverse transition probability matrix.

Now by Markov renewal equation which holds for all states i,j=1,2,…m can be put as

$F_{ij}(t) = a_{ij}(t) + \int_0^t \sum_k C_{ik}(x) f_{kj}(t-x)dx,\ i,j = 1,2, \ldots .m$(2)

and in matrix form as

$F(t) = A(t) + \int_0^t C(x)F(t-x)dx$ (3)

Denoting the convolution of C and F by $C * F$, we can put the same as
$F(.) = A(.) + C(.) * F(.)$ (4)
Taking Laplace transforms we get
$F * (s) = A * (s) + C * (s). F * (s)$ (5)
where C is a somewhat special type of matrix.

III.CONCLUSION

In this paper we presented a new approximation method for characterizing the superposition of multiple independent Markov Renewal processes. The presentation here focused on discrete-time processes, but the methodology is readily applicable to continuous-time processes with little modification. One special case of this model is the arbitrary on/off source that was used in this paper. We have presented a queuing model for the analysis of a statistical multiplexer whose input is a MRP representing the superposition of multiple traffic sources.

The advantage of our methodology is that it provides a uniform frame work in which a variety of models of traffic sources can be handled. The basic limitation is the huge sale space and the computational complexity of the algorithms. These disadvantages can be overcome if an elegant state aggregation scheme is found such that the resulting aggregate process has a number of states which is preferably a linear function of the number of sources while still preserving the characteristics of the real superposition process. We introduced an aggregation scheme for reducing the dimensionality of the superposition process. However, during the aggregation, the statistical properties of the original process may be distorted and may subsequently lead to inaccurate results for the multiplexer's performance metrics.

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