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A Fixed Point Result for Cyclic Mapping in Extended B-Metric Spaces

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Abstract
In an extended b-metric space, the properties of solutions of fixed point equations of LS-type cyclic mapping are discussed. In this paper, we establish the fixed point theorem on LS-type cyclic mapping in an extended b-metric space. Our results extend some known conclusions in the literature.
Keywords: *Fixed point theory, extended b-metric spaces, LS-type cyclic mapping*
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I. Introduction

Fixed point theory is a vigorous research field, which has important applications in dynamic systems, functional equations, differential equations, economics and so on. With the development of mathematics, the application of fixed-point theory is also expanding, which provides powerful tools and methods for the research of various disciplines. Fixed point theory is also applied to the study of functional equations and differential equations. The existence and uniqueness of a given equation can be solved by finding the fixed point of the equation. This is of great significance for mathematical modeling and analysis. In addition, the fixed-point theory has been widely used in economics, physics and other disciplines. In economics, the equilibrium state and stability of an economic system can be studied by constructing a fixed point of an economic model. In physics, fixed point theory is applied to describe phase transition processes and the properties of equilibrium states. Metric fixed-point theory is a branch of fixed-point theory that deals with the properties of solutions to fixed point equations of operators in extended metric spaces. In 2017, the scholars from Pakistan, Tayyab Kamran, Maria Samreen and Qurat UL Ain [1] proposed the concept of extended b-metric space based on b-metric space [2], they finish the work of generalization of b-metric space and extend Banach contraction mapping principle [3] into in extended b-metric space, obtained the corresponding fixed-point theorems. For b-metric space, more references, interested readers may refer to the literatures [4,5]. For extended b-metric space, more references, interested readers may refer to the literatures [6,7].Therefore, in this paper, we consider a known cyclic mapping, which is found in [8], and then explore the properties of the fixed-point equation of the cyclic mapping in an extended b-metric space.

II. Preliminary Knowledge

In this section, some basic information about the extended b-metric space are given in the following content.

Definition 1 Let X be a nonempty set and $\theta: X \times X \to [1, \infty)$. A function $d_\theta: X \times X \to [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$ it satisfies:

($d_\theta 1$) $d_\theta(x, y) = 0$ if and only if $x = y$;
($d_\theta 2$) $d_\theta(x, y) = d_\theta(y, x)$;
($d_\theta 3$) $d_\theta(x, y) \le \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)]$.

The pair (X, d_θ) is called an extended b-metric space.

Next, we present some concrete examples about an extended b-metric space.

Example 1 Let $X = \{1,2,3\}$. Define $\theta: X \times X \to [1, \infty)$, $d_\theta: X \times X \to [0, \infty)$ as follows:

$$\theta(x, y) = 1 + x + y,\ d_\theta(1,1) = d_\theta(2,2) = d_\theta(3,3) = 0$$
$$d_\theta(1,2) = d_\theta(2,1) = 80,\ d_\theta(1,3) = d_\theta(3,1) = 1000,\ d_\theta(3,2) = d_\theta(2,3) = 600$$

The metric space defined by the above has been proved to be an extended b-metric space in reference [1].

Example 2 Let $X = [0, \infty)$. Define $\theta: X \times X \to [1, \infty)$, $d_\theta: X \times X \to [0, \infty)$ as follows:

$$\theta(x, y) = 2 + x + y,\ d_\theta(x, y) = (x - y)^2$$

Then (X, d_θ) is a complete extended b-metric space [1].

Next, the concepts of convergence, Cauchy sequence and completeness will be given in an extended b-metric space.

Definition 2 Let (X, d_θ) be an extended b-metric space.
(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \bar{N}$ such that $d_\theta(x_n, x) < \varepsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \bar{N}$ such that $d_\theta(x_n, x_m) < \varepsilon$ for all $m, n \geq N$.
Definition 3 An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.
In the sequel, the notion of a closed set will be used; therefore, we present it here.
Definition 4 Let (X, d_θ) an extended b-metric space, $C \subseteq X$ be a nonempty set. The set C is closed, if for $\forall \{x_n\} \subseteq C$, there exists a$x^*$ such that $x_n \to x^*$ as $n \to \infty$, then $x^* \in C$.
Definition 5 Let G_1, G_2 be nonempty sets of metric space, if $B(G_1) \subseteq G_2$, $S(G_2) \subseteq G_1$, thus the (B, S): $G_1 \times G_2 \to G_2 \times G_1$ is called as a pair semi-cyclic mapping, where B is said to be a lower semi-cyclic mapping, S is said to be an upper semi-cyclic mapping. If $B = S$, then B is said to be a cyclic mapping.

III. Fixed Point Theorems

This section first gives the definition of *LS*-type cyclic mapping, and then gives the main conclusions of this paper.
Definition 6 [8] Suppose that G_1, G_2 are two nonempty sets in an extended b-metric space (X, d_θ) and if (B,S) is a pair semi-cyclic mapping on $G_1 \times G_2$ and there exists γ, δ, t, L and γ, δ, t, L are some real nonnegative numbers such that for all $x \in$ Q1, $y \in$ Q2, the following inequality holds:

$$\gamma d(u, Bu) + \delta d(v, Sv) + td(Bu, Sv) - d(u, Sv) \leq Ld(u, v)$$

Then, the pair (B, S) is called a *LS*-type cyclic mapping.

Next, the main conclusions about the existence and uniqueness of fixed points of LS type cyclic mapping are given based on an extended b-metric space.

Theorem 1 Assume that (X, d_θ) is a complete extended b-metric space, and the pair (B, S) is the *LS*-type cyclic mapping in (X, d_θ). Let G_1, G_2 be two nonempty closed subsets in extended b-metric space (X, d_θ), $G_1 \cap G_2 \neq \varnothing$, if the following conditions hold:

(I) $\delta > 1, L > \delta, L + \theta(x_{2n}, x_{2n+2}) > \gamma;$

(II) $L + 2\theta(x_{2n}, x_{2n+2}) < \delta + t + \gamma, L + 1 \leq t;$

(III) $\limsup_{n\to\infty} \theta(x_{2n}, x_{2n+2}) < 1/h, h = \max\left\{\frac{L + \theta(x_{2n}, x_{2n+2}) - \gamma}{\delta + t - \theta(x_{2n}, x_{2n+2})}, \frac{L - \delta}{\gamma + t}\right\}.$

Then, there exists a unique point $x^* \in G_1 \cap G_2$ such that $Bx^* = x^* = Sx^*$.

Proof of Theorem: Firstly define $\{x_n\}$ and let $x_0 \in G_1$ as follows:

$$x_1 = Bx_0, x_2 = Sx_1, x_3 = Bx_2, x_4 = Sx_3, \cdots, x_{2n+1} = Bx_{2n}, x_{2n+2} = Sx_{2n+1}, \cdots, n \geq 0$$

Since the pair (B, S) is a *LS*-type cyclic mapping, from Definition 6, it can get

$$\begin{aligned}
Ld_\theta(x_0, x_1) &\geq \gamma d_\theta(x_0, Bx_0) + \delta d_\theta(x_1, Sx_1) + td_\theta(Bx_0, Sx_1) - d_\theta(x_0, Sx_1) \\
&= \gamma d_\theta(x_0, x_1) + \delta d_\theta(x_1, x_2) + td_\theta(x_1, x_2) - d_\theta(x_0, x_2) \\
&\geq \gamma d_\theta(x_0, x_1) + \delta d_\theta(x_1, x_2) + td_\theta(x_1, x_2) \\
&\quad - \theta(x_0, x_2)\left[d_\theta(x_0, x_1) + d_\theta(x_1, x_2)\right] \\
&= \gamma d_\theta(x_0, x_1) + \delta d_\theta(x_1, x_2) + td_\theta(x_1, x_2) \\
&\quad - \theta(x_0, x_2) d_\theta(x_0, x_1) - \theta(x_0, x_1) d_\theta(x_1, x_2)
\end{aligned} \tag{1}$$

Then

$$\left(L + \theta(x_0, x_1) - \gamma\right) d_\theta(x_0, x_1) \geq \left(\delta + t - \theta(x_0, x_1)\right) d_\theta(x_1, x_2)$$

That is

$$d_\theta\left(x_1,x_2\right)\leq\frac{L+\theta\left(x_0,x_2\right)-\gamma}{\delta+t-\theta\left(x_0,x_2\right)}d_\theta\left(x_0,x_1\right)$$

Next, we continue to use the definition of *LS*-type cyclic mapping, and we can obtain

$$\begin{aligned}Ld_\theta\left(x_2,x_1\right)&\geq\gamma d_\theta\left(x_2,Bx_2\right)+\delta d_\theta\left(x_1,Sx_1\right)+td_\theta\left(Bx_2,Sx_1\right)-d_\theta\left(x_2,Sx_1\right)\\&=\gamma d_\theta\left(x_2,x_3\right)+\delta d_\theta\left(x_1,x_2\right)+td_\theta\left(x_3,x_2\right)-d_\theta\left(x_2,x_2\right)\\&=\gamma d_\theta\left(x_2,x_3\right)+\delta d_\theta\left(x_1,x_2\right)+td_\theta\left(x_3,x_2\right)\end{aligned}\qquad(2)$$

Sort it out and this implies

$$\left(L-\delta\right)d_\theta\left(x_1,x_2\right)\geq\left(\gamma+t\right)d_\theta\left(x_2,x_3\right)$$

That is

$$d_\theta\left(x_2,x_3\right)\leq\frac{L-\delta}{\gamma+t}d_\theta\left(x_0,x_1\right)$$

The same process (1)-(2), we're doing it again and get

$$\begin{aligned}Ld_\theta\left(x_2,x_3\right)&\geq\gamma d_\theta\left(x_2,Bx_2\right)+\delta d_\theta\left(x_3,Sx_3\right)+td_\theta\left(Bx_2,Sx_3\right)-d_\theta\left(x_2,Sx_3\right)\\&=\gamma d_\theta\left(x_2,x_3\right)+\delta d_\theta\left(x_3,x_4\right)+td_\theta\left(x_3,x_4\right)-d_\theta\left(x_2,x_4\right)\\&\geq\gamma d_\theta\left(x_2,x_3\right)+\delta d_\theta\left(x_2,x_4\right)+td_\theta\left(x_3,x_4\right)\\&\quad-\theta\left(x_2,x_4\right)\left[d_\theta\left(x_2,x_3\right)+d_\theta\left(x_3,x_4\right)\right]\\&=\gamma d_\theta\left(x_2,x_3\right)+\delta d_\theta\left(x_2,x_4\right)+td_\theta\left(x_3,x_4\right)\\&\quad-\theta\left(x_2,x_4\right)d_\theta\left(x_2,x_3\right)-\theta\left(x_0,x_1\right)d_\theta\left(x_3,x_4\right)\end{aligned}$$

Then we have

$$d_\theta\left(x_3,x_4\right)\leq\frac{L+\theta\left(x_2,x_4\right)-\gamma}{\delta+t-\theta\left(x_2,x_4\right)}d_\theta\left(x_2,x_3\right)$$

And

$$\begin{aligned}Ld_\theta\left(x_4,x_3\right)&\geq\gamma d_\theta\left(x_4,Bx_4\right)+\delta d_\theta\left(x_3,Sx_3\right)+td_\theta\left(Bx_4,Sx_3\right)-d_\theta\left(x_4,Sx_3\right)\\&=\gamma d_\theta\left(x_4,x_5\right)+\delta d_\theta\left(x_3,x_4\right)+td_\theta\left(x_5,x_4\right)-d_\theta\left(x_4,x_4\right)\\&=\gamma d_\theta\left(x_4,x_5\right)+\delta d_\theta\left(x_3,x_4\right)+td_\theta\left(x_5,x_4\right)\end{aligned}$$

Thus, we get

$$d_\theta\left(x_4,x_5\right)\leq\frac{L-\delta}{\gamma+t}d_\theta\left(x_3,x_4\right)$$

It can derive from conditions (I)-(II) that

Let $h=\max\left\{\frac{L+\theta\left(x_{2n},x_{2n+2}\right)-\gamma}{\delta+t-\theta\left(x_{2n},x_{2n+2}\right)},\frac{L-\delta}{\gamma+t}\right\}$, then we can know that $h\in\left(0,1\right)$.

So, we can get

$$d_\theta\left(x_1,x_2\right)\leq hd_\theta\left(x_0,x_1\right)\text{ and }d_\theta\left(x_2,x_3\right)\leq hd_\theta\left(x_1,x_2\right)\leq h^2d_\theta\left(x_0,x_1\right)\qquad(3)$$

Continue the above process (1)-(3), we have

$$d_\theta\left(x_n,x_{n+1}\right)\leq h^nd_\theta\left(x_0,x_1\right)$$

With the help of condition (IV), $\forall m,n\in N$ and $n\leq m$, then

$$\begin{aligned}
d_\theta(x_n,x_m) &\le \theta(x_n,x_m)\left[d_\theta(x_n,x_{n+1})+d_\theta(x_{n+1},x_m)\right]\\
&\le \theta(x_n,x_m)d_\theta(x_n,x_{n+1})\\
&\quad +\theta(x_n,x_m)\theta(x_{n+1},x_m)\left[d_\theta(x_{n+1},x_{n+2})+d_\theta(x_{n+2},x_m)\right]\\
&= \theta(x_n,x_m)d_\theta(x_n,x_{n+1})+\theta(x_n,x_m)\theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2})\\
&\quad +\theta(x_n,x_m)\theta(x_{n+1},x_m)\theta(x_{n+2},x_m)d_\theta(x_{n+2},x_m)\\
&\le \theta(x_n,x_m)d_\theta(x_n,x_{n+1})+\theta(x_n,x_m)\theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2})+\ldots\\
&\quad +\theta(x_n,x_m)\theta(x_{n+1},x_m)\ldots\theta(x_{m-1},x_m)d_\theta(x_{m-1},x_m)\\
&\le \theta(x_n,x_m)h^n d_\theta(x_0,x_1)+\theta(x_n,x_m)\theta(x_{n+1},x_m)h^{n+1}d_\theta(x_0,x_1)+\ldots\\
&\quad +\theta(x_n,x_m)\theta(x_{n+1},x_m)\ldots\theta(x_{m-1},x_m)h^{m-1}d_\theta(x_0,x_1)
\end{aligned} \tag{4}$$

With aid of condition (III) and $\limsup\limits_{n\to\infty}\theta(x_{2n},x_{2n+2})<1/h$, it can derive from inequality (4) that the series $\sum_{n=1}^{\infty}h^n\prod_{i=1}^{n}\theta(x_i,x_m)$ converges by ratio test for each $m\in N$. Let

$$E=\sum_{n=1}^{\infty}h^n\prod_{i=1}^{n}\theta(x_i,x_m),E_n=\sum_{j=1}^{n}h^j\prod_{i=1}^{j}\theta(x_i,x_m)$$

For $m>n$, the above inequality can show the following result

$$d_\theta(x_n,x_m)\le d_\theta(x_0,x_1)(E_{m-1}-E_n)$$

Let $n\to+\infty$, this can get

$$\lim_{n\to\infty}d_\theta(x_m,x_n)=0 \tag{5}$$

Therefore, it can derive from (5) that the sequence $\{x_n\}$ is a Cauchy sequence.

Since (X,d_θ) is a complete extended b-metric space，then there exists a $x\in X$ such that $x_n\to x(n\to\infty)$.

Thus, we can get $x_{2n}\to x; x_{2n+1}\to x(n\to\infty)$.

Since $\{x_{2n}\}\subseteq G_1,\{x_{2n+1}\}\subseteq G_2$, and G_1,G_2 are two nonempty closed sets, then $x\in G_1\bigcap G_2$. Next we show that the point $x\in G_1\bigcap G_2$ such that $Bx=x=Sx$. Since the pair (B,S) is *LS*-type cyclic mapping, thus we can obtain

$$Ld_\theta(x,x)\ge\gamma d_\theta(x,Bx)+\delta d_\theta(x,Sx)+td_\theta(Bx,Sx)-d_\theta(x,Sx)$$

That is

$$0\ge\gamma d_\theta(x,Bx)+(\delta-1)d_\theta(x,Sx)+td_\theta(Bx,Sx)$$

Since $\gamma\ge0,\delta>1,t\ge0$, thus, it can be obtained that

$$d_\theta(x,Bx)=0,d_\theta(x,Sx)=0,d_\theta(Bx,Sx)=0$$

That is

$$Bx=x=Sx$$

Here the purpose is to show that $x\in G_1\bigcap G_2$ is a unique point such that $Bx=x=Sx$. Suppose that $x,x^*\in X$ are the common fixed point of *B* and *S*，Then utilizing the Definition 6， we can get

$$Ld_\theta(x,x^*)\ge\gamma d_\theta(x,Bx)+\delta d_\theta(x^*,Sx^*)+td_\theta(Bx,Sx^*)-d_\theta(x,Sx^*)$$

That is

$$Ld_\theta(x,x^*)\ge\gamma d_\theta(x,x)+\delta d_\theta(x^*,x^*)+td_\theta(x,x^*)-d_\theta(x,x^*)$$

Thus, we can obtain

$$Ld_\theta(x,x^*)\ge(t-1)d_\theta(x,x^*)$$

It can derive from condition $L+1 \le t$ that

$$d_\theta\left(x, x^*\right)=0$$

This completes the proof of this results.

Remark 1 In this paper, the properties of the solution of the fixed-point equation (*Bx=x* and *Sx=x*) for *LS*-type cyclic mapping are investigated, and the existence and uniqueness of the common fixed point for the pair *B* and *S* are obtained successfully.

IV. Conclusions

The metric fixed point theory is interesting. Constructing mapping equipped with different structures in extended metric space and then studying the properties of solutions of their fixed- point equation is also an important research work in this branch of research. Based on an extended b-metric space, this paper probes into the properties of the solution of *LS*-type cyclic mapping's fixed point equation, and successfully establishes the corresponding fixed point theorem. As for the extension of the follow-up work of this article, we can also consider extending to some other spaces, such as, rectangular metric space [9], rectangular b-metric space [10] and so on.

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References

[1]. T. Kamran., M. Samreen., Q. UL Ain. (2017). A Generalization of b-Metric Space and Some Fixed Point Theorems. Mathematics, **2017**, 5, 19; doi:10.3390/math5020019.

[2]. H. Aydi., M.-F. Bota., E. Karapınar., S. Mitrovic. (2018). A fixed point theorem for set-valued quasi-contractions in b-metric spaces, Fixed Point Theory and Applications, 2012, Article ID 88, 2012.

[3]. S. Czerwik. (1993). Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, 1, 5-11, 1993.

[4]. B. A. S. Alamri., R. P. Agarwal., J. Ahmad., (2018) .Some New Fixed Point Theorems in b-Metric Spaces with Application, Mathematics, 8(5), 725, 2018. https://doi.org/10.3390/math8050725.

[5]. S. Banach.(1922). Surles operations dans les ensembles abstraits et leur applications aux equations integrals[J]. Fundam Math, 1922, 3(1): 133-181. doi:10.4064/FM-3-1-133-181 Corpus ID: 118543265.

[6]. S.Q.Weng, X. LI. (2024). Fixed Point Theorem of LS-type Cyclic Mapping in b-Metric Spaces. Journal of Yibin University. 2024, 24 (06). Accepted for publication.

[7]. H.P. Huang, Yumnam Mahendra Singh.; Mohammad Saeed Khan.; Stojan Radenovic. (2021) Rational Type Contractions in Extended b-Metric Spaces.Symmetry. 2021, 13(4), 614; https://doi.org/10.3390/sym13040614.

[8]. L.Guran, M.F.Bota. (2021) Existence of the Solutions of Nonlinear Fractional Differential Equations Using the Fixed Point Technique in Extended b-Metric Spaces. Symmetry .2021, 13(2), 158; https://doi.org/10.3390/sym13020158.

[9]. A. Branciari..(2000). A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces.Publ. Math. Debrecen. 2000, 57 (1-2), 31–37.

[10]. R. Georgea, S. Radenovicb, K.P. Reshmac, S. Shukla. (2015). Rectangular b-metric space and contraction principles. J. Nonlinear Sci. Appl. 2015, 8, 1005-1013.