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Secure Domination and Independent Domination number of Knot Product Graphs

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Abstract
Domination is one of the remarkable area in graph theory. It has applications in Biology, Security, Networks. In this article Secure Domination and Independent Domination number of Knot Product graph of few graphs are calculated.
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I. INTRODUCTION

Domination is one of the notable area in graph theory. The concept of domination was used by De Jaenisch in 1862 [3] while studying problems of determining the minimum number of queens to dominate Chess Board. In 1958 Claude Berge [2] defined the concept of domination number of graph. In 1962 Oystein Ore [9] introduced the terms "dominating set" and "domination number" in his book on graph theory.

Security in graphs was first defined by Brigham, Dutton and Hedetniemi in 2007 [4]. Set of all the vertices adjacent to a vertex v in $V(G)$ is called neighbourhood of v and is denoted by $N(v)$. Closed neighbourhood of vertex v in $V(G)$ is denoted by $N[v]$ and is defined as $N[v] = N(v) \cup \{v\}$. A set S ⊆ $V(G)$ is said to be secure if the security condition, for every X ⊆ S, N[X] ∩ S ≥ N[X] - S holds. Now, a set S ⊆ $V(G)$ is secure dominating if it is both secure and dominating. The secure domination number of G is the minimum cardinality of a secure dominating set in G and is denoted by $\gamma_s(G)$. The theory of Independent Domination was formalized by Berge and Ore in 1962. The Independent domination number notation was introduced by Cockayne and Hedetniemi [6]. A set S of vertices in a graph G is called an Independent dominating set if S is both independent and dominating. The Independent Domination number of G is the minimum cardinality of an independent dominating set in G and is denoted by $\gamma_i(G)$.

Knot product graph was introduced by B. Basavanagouda and Keerthi G. Mirajkar [1]. Knot product graph $G \otimes H$ of two graphs G and H with vertex set $V(G) X\, V(H)$ is defined as any two points (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1 = u_2$ or [u_1 is adjacent to u_2 and v_1 is adjacent to v_2].

Graph $G = (V, E)$ be finite, simple graphs of order n. A subset D of V is dominating set if every vertex in V-D is adjacent to atleast one vertex of D. Domination number is the minimum cardinality of dominating set and is denoted by $\gamma(G)$. A u-v walk of graph G is an alternating sequence of vertices and edges of G. Path is a walk if all it's vertices are distinct. Path of length n is denoted by P_n. Cycle is a non-trivial closed path. Cycle of length n is denoted as C_n. Complete graph K_n is a graph in which every vertex is adjacent to remaining all the vertices. In this article Secure Domination and Independent Domination number of Knot Product of graph of paths, cycles and complete graphs are calculated. For undefined terminologies refer [7,11,12]. To know more on domination refer [5,6,8,10,13].

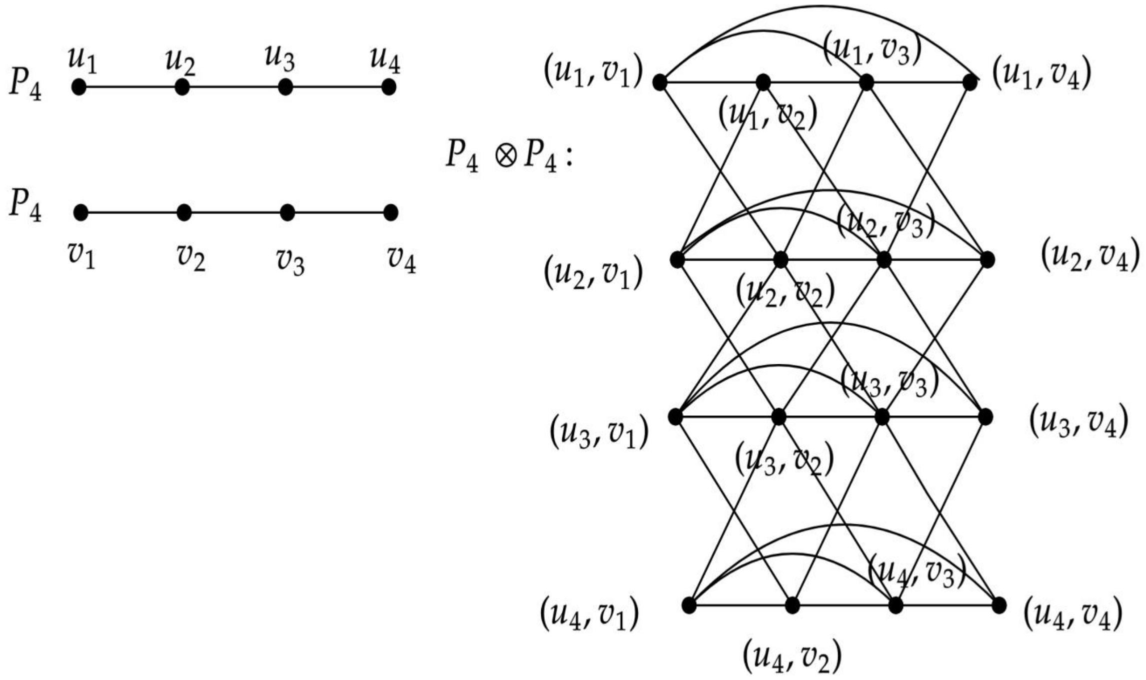

Figure 1. Knot product graph of $P_4 \otimes P_4$.

The following are of immediate use.

Remark 1: [5] Secure Domination of Complete graph is [n/2].

II. Main Results

Here, Secure Domination and Independent Domination number of Knot Product graph of paths, cycles and complete graphs is calculated.

Theorem 2.1. The Secure Domination number of Knot Product graph of P_1 and G is,

$$\gamma_s(P_1 \otimes G) = [n/2].$$

Proof. Consider path P_1 with vertex u_1 and G containing $v_1, v_2, \ldots, v_n$ as vertices. The vertex set of Knot Product grpah $P_1 \otimes G$ contains u_1 as first term in all ordered pairs. From the definition of Knot Product graph all vertices are adjacent to one another. Hence the domination number is 1. But in case of secure domination consider both secure and dominating set. From definition of Knot Product the vertices $(u_1, v_1), (u_1, v_2), (u_1, v_3), \ldots, (u_1, v_n)$ all are connected to one another because they have same x co-ordinate. Hence, Knot Product graph of P_1 and G forms complete graph and from Remark 1, domination number of complete graph is [n/2].

Hence,

$$\gamma_s(P_1 \otimes G) = [n/2].$$

□

Theorem 2.2. The Secure Domination number of Knot Product graph of G and P_1 is,

$$\gamma_s\,(G \otimes P_1) = n.$$

Proof. Consider path P_1 with vertex u_1 and G be the graph having $v_1, v_2, \ldots, v_n$ as vertices. The Knot Product graph $G \otimes P_1$ contains vertices $(v_1, u_1), (v_2, u_1), \ldots, (v_\mathrm{n}, u_1)$. From definition of Knot Product graph the vertex (v_1, u_1) is not connected to any other vertices of $G \otimes P_1$. Similarly $(v_2, u_1), (v_3, u_1), \ldots, (v_\mathrm{n}, u_1)$ are also not connected to any other vertices of $G \otimes P_1$. Hence, Knot Product of G and P_1 forms the disconnected graph. Since Knot Product of G and P_1 has n vertices the secure domination will be n.

Therefore,

$$\gamma_s(G \otimes P_1) = n.$$

□

Theorem 2.3. For any graph G,

$$\gamma_s(\mathrm{G} \otimes P_1) = \gamma_s(P_1 \otimes \mathrm{G}), \text{ if } [n/2]= n \text{ and } \forall\, n \geq 2.$$

Proof. Consider path P_1 and G.

Let $V(P_1) = \{u_1\}$ and $V(\mathrm{G}) = \{v_1, v_2, \ldots, v_n\}$.

From Theorem 2.1,

$$\gamma_s(P_1 \otimes G) = [\mathrm{n}/2].$$

From Theorem 2.2,

$$\gamma_s(G \otimes P_1) = n.$$

Further, the Knot Product of secure domination numbers of G $\otimes$ P_1 and $P_1 \otimes$ G are equal at the points where n is equal to greatest integer function of n/2.

Hence,

$$\gamma_s(\mathrm{G} \otimes P_1) = \gamma_s(P_1 \otimes \mathrm{G}), \ \forall n \geq 2, \text{ if } [n/2]= n.$$

□

Theorem 2.4. The Independent Domination number of Knot product graph of P_m and P_n is,

$$\gamma_i(P_n \otimes P_m) = n, \quad \forall m \neq 2, \quad m, n \in N.$$

Proof. Consider the paths P_n and P_m. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n and $v_1, v_2, \ldots, v_m$ represents the vertices of P_m . The vertex set of Knot Product graph of P_n and P_m contains ordered pairs

$$(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_m),$$

$$(u_2, v_1), (u_2, v_2), \ldots, (u_2, v_m),$$

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$(u_n, v_1), (u_n, v_2), \ldots, (u_n, v_m)$ as vertices.

To prove the result we consider the method of Mathematical Induction.

For $n = 1$,

$$\gamma_i(P_1 \otimes P_m) = 1.$$

Because, the vertex (u_1,v_1) forms the edge with $(u_1,v_2),(u_1,v_3), \ldots ,(u_1,v_n)$.

Hence, it is true for $n = 1$.

For $n = 2$,

$$\gamma_i(P_2 \otimes P_m) = 2\ .$$

Because, the vertices of $(P_2 \otimes P_n)$ dominates the vertices (u_1, v_1) and (u_2, v_1)

Hence, it is true for $n = 2$.

Now, assume that $\gamma_i(P_n \otimes P_m) =$ n is true $\forall m \neq 2,\quad m, n \in N$

Consider, the Knot Product graph ($P_{n+1} \otimes P_m$). The vertex set of $(P_{n+1} \otimes P_m)$ contains ordered pairs whose x co-ordinate belong to V(P_{n+1}) and y co-ordinate belong to V(P_m). (u_1, v_1) dominate all the vertices whose first term of ordered pair is u_1. (u_2, v_1) dominate those vertices whose first term of the ordered pair is u_2. In same way it is true for other vertices. Hence the number of vertices which dominate graph is $n + 1$. By the method of mathematical induction the result is true for n.

Hence,

$$\gamma_i(P_n \otimes P_m) = n, \quad \forall m \neq 2, \quad m, n \in N.$$

□

Theorem 2.5. The Independent Domination number of Knot Product graph of path P_n and G is,

$$\gamma_i(P_n \otimes G) = n, \quad \forall n \succ 3, \text{ where } n \in N.$$

Proof. Let $u_1, u_2, \ldots, u_n$ represent the vertices of $P_n$and Let G contain $v_1, v_2, \ldots, v_m$ as vertices. The vertex set of Knot Product graph of P_n and G as follows,

$(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_m)$,

$(u_2, v_1), (u_2, v_2), \ldots, (u_2, v_m)$,

. . . .

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$(u_n, v_1), (u_n, v_2), \ldots, (u_n, v_m)$.

The vertex (u_1, v_1) forms an edge with all the vertices whose first term of ordered pair is u_1. Hence,(u_1, v_1)dominate the vertices whose x co-ordinate is u_1.Similarly $(u_2, v_1), (u_3, v_1), \ldots, (u_m, v_1)$ dominate remaining vertices of graph $P_n \otimes G$.

Therefore,

$$\gamma_i(P_n \otimes G) = n\,, \ \ \forall n \succ 3.$$

□

Theorem 2.6. The Independent domination number of Knot Product graph of any graph G and P_1 is,

$$\gamma_i(G \otimes P_1) = m.$$

Proof. Let G be a graph of order m. Consider path P_1 with vertex u_1 and G be any graph with vertices $v_1, v_2, \ldots, v_{m.}$ The Knot Product graph $G \otimes P_1$ contains vertices $(v_1, u_1), (v_2, u_1), \ldots, (v_n, u_1)$. From the definition of Knot Product of graph the vertex (v_1, u_1) is not connected to any other vertices of $(G \otimes P_1)$. Similarly, $(v_2, u_1), (v_3, u_1), \ldots, (v_n, u_1)$ are also not connected to any other vertices of $(G \otimes P_1)$. Hence, Knot Product of G and P_1 forms the disconnected graph with m vertices.

Therefore,

$$\gamma_i(G \otimes P_1) = m.$$

□

Theorem 2.7. For paths P_n *and* P_m,

$$\gamma_i(P_m \otimes P_n) = \gamma_i(P_n \otimes P_m), \quad if \quad n = m, \quad \forall n, m \succ 2.$$

Proof. Consider P_m and P_n

Let $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$

From Theorem 2.4 ,

$$\gamma_i(P_m \otimes P_n) = m$$

$$\gamma_i(P_n \otimes P_m) = n$$

Therefore,

$$\gamma_i(P_m \otimes P_n) = \gamma_i(P_n \otimes P_m), \ \ if \quad n = m, \ \ \forall n, m \succ 2.$$

□

Theorem 2.8. The Independent Domination number of Knot Product graph of C_m and C_n is,

$$\gamma_i(C_m \otimes C_n) = m, \quad \forall m, n \geq 3.$$

Proof. Consider the cycles C_n and C_m .

Let $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$

Vertices of the Knot Product graph of C_m and C_n are classified into n different sets based on their internal connections ,

$$\{(u_1, v_1), (u_1, v_2), \dots, (u_1, v_n)\}$$

$$\{(u_2, v_1), (u_2, v_2), \dots, (u_2, v_n)\}$$

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$$\{(u_m, v_1), (u_m, v_2), \dots, (u_m, v_n)\}.$$

Here, the vertices in each set are adjacent to each another, if one vertex is considered from each set those vertices will be acting as a dominating vertex. Total number of dominating vertices will be m.

Therefore,

$$\gamma_i(C_m \otimes C_n) = m, \ \ \forall m, n \geq 3.$$

□

Theorem 2.9. The Independent Domination number of Knot Product graph of $\mathrm{P_m}$ and $\mathrm{C_n}$ is,

$$\gamma_i(P_m \otimes C_n) = m, \quad \forall n \geq 3.$$

Proof. Let P_m and C_n be the path and cycle of lengths m and n respectively.

Let $V(P_m) = \{u_1, u_2, \dots, u_m\}$ and $V(C_n) = \{v_1, v_2, \dots, v_n\}$

The Knot Product graph of P_m and C_n contains vertices whose x co-ordinate Corresponds to $V(P_m)$ and Y co-ordinate Corresponds to $V(C_n)$. The vertices $(u_1, v_n), (u_2, v_n), \dots, (u_m, v_n)$ dominate the vertices of Knot Product of P_m and C_n.

Therefore,

$$\gamma_i(P_m \otimes C_n) = m, \quad \forall n \geq 3.$$

□

Theorem 2.10. The Independent Domination number of Knot Product graph of K_m and K_n is,

$$\gamma_i(K_m \otimes K_n) = m, \quad m, n \in N.$$

Proof. Consider complete graphs K_m and K_n. K_m contains $u_1, u_2, u_3, \dots, u_m$ as vertices. K_n contains $v_1, v_2, v_3, \dots, v_n$ as vertices. The vertex set of Knot Product graph of K_m and K_n contains,

$$(u_1,v_1), (u_1,v_2), \dots, (u_1,v_n),$$

$$(u_2,v_1), (u_2,v_2), \dots, (u_2,v_n),$$

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$(u_m,v_1), (u_m,v_2), \dots ,(u_m,v_n)$.

We consider the Induction method to construct the Proof.

For m=1,

$$\gamma_i\ (K_1 \otimes K_n) = 1.$$

Because, the vertex (u_1,v_1) is adjacent to the vertices $(u_1,v_2), (u_1,v_3), \dots , (u_1,v_n)$.

Hence, it is true for m=1 .

For m=2 ,

$$\gamma_i\ (K_2 \otimes K_n) = 2 .$$

Because, the vertices of Knot product of K_2 and K_n are dominated by the vertices (u_1,v_1) and (u_2,v_1).

Hence, it is true for m=2 .

Now , assume $\gamma_i(K_m \otimes K_n) = m$ is true for m.

consider Knot Product graph K_{m+1} and K_n . The first term of ordered pair of vertices is from V (K_{m+1}) and second term of ordered pair of vertices is from $V(K_n)$. The vertex (u_1,v_1) dominate those vertices whose first term of order pair is u_1. (u_2,v_1) dominate those vertices whose first term of ordered pair is u_2. Similarly, it holds for other vertices. Hence, the number of vertices which dominate entire graph is m+1. Therefore, from mathematical induction the result is true for m.

$$\gamma(K_m \otimes K_n) = m\ ,\ \ m,n \in N.$$

□

III. CONCLUSION

Domination number has applications in Security, Biology. In this article the Secure Domination and Independent domination number of Knot Product graph of paths, cycles, and complete graphs are calculated.

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