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An Application of Carleson Measures on Simply Connected Domains

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Abstract.
M. J. Gonz´alez [Go] study the Carleson measures based and associated to the Hardy andweighted Bergman spaces defined on general simply connected domains. This program was initiated byZinsmeister in his paper Les domaines de Carleson (1989), where he shows that the geometry of thedomain plays a fundamental role. We will review the classical results presenting, in some cases, alternative proofs and will examine analogously the situation for the weighted Bergman spaces.
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I. Introduction

For $\mathbb{D}$ denote the unit disc $\{z \in \mathbb{C}; |z| < 1\}$ and let $\mathbb{T} = \partial\mathbb{D}$. If $I \subset \mathbb{T}$ is an interval, the Carleson square $S(I) \subset \mathbb{D}$ is the set

$$S(I) = \left\{(1+\epsilon)e^{i\theta_j} : e^{i\theta_j} \in I, -\frac{|I|}{2\pi} \le \epsilon < 0\right\}$$

where $|I|$ denotes the length of the interval I.
We will consider the Hardy spaces and the weighted Bergman spaces. A sequence of an analytic functions f_j in $\mathbb{D}$ is in the Hardy space $\mathcal{H}^{1+\epsilon}, 0 \le \epsilon < \infty$, if

$$\sup_{0<\epsilon<1} \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j\left((1-\epsilon)e^{i\theta_j}\right)\right|^{1+\epsilon} d\theta_j = \sum_j \| f_j \|_{\mathcal{H}^{1+\epsilon}}^{1+\epsilon} < \infty$$

The functions $f_j \in \mathcal{H}^{1+\epsilon}$ has almost everywhere non-tangential boundary limit $f_j\left(e^{i\theta_j}\right) \in L^{1+\epsilon}(\mathbb{T})$, and $\| f_j \|_{\mathcal{H}^{1+\epsilon}}^{1+\epsilon} = \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j\left(e^{i\theta_j}\right)\right|^{1+\epsilon} d\theta_j$.
The weighted Bergman spaces $\mathcal{A}_{\epsilon-1}^{1+\epsilon}, 0 \le \epsilon < \infty$, consists of those analytic functions f_j in $\mathbb{D}$ such that

$$\frac{1}{\pi}\int_{\mathbb{D}} \sum_j |f_j(z)|^{1+\epsilon}(1-|z|)^{\epsilon-1} dm(z) = \sum_j \| f_j \|_{\mathcal{A}_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} < \infty$$

where dm is the Lebesgue area measure. When $\epsilon = 1$ we obtain the classical Bergman spaces. See [D], [G] and [HKZ] for Hardy and Bergman spaces.
A known theorem due to Carleson [C] shows that a positive measure μ defined in $\mathbb{D}$ gives for all $0 \le \epsilon < \infty$,

$$\int_{\mathbb{D}} \sum_j |f_j(z)|^{1+\epsilon} d\mu(z) \le C_{1+\epsilon} \sum_j \| f_j \|_{\mathcal{H}^{1+\epsilon}}^{1+\epsilon}, \text{ for all } f_j \in \mathcal{H}^{1+\epsilon}$$

if and only if for $\epsilon \ge 0$ and all intervals $I \subset \mathbb{T}, \mu(S(I)) < (1+\epsilon)|I|$.
The known theorem has led to various generalizations, including analogous results for the weighted Bergman spaces $\mathcal{A}_{\epsilon-1}^{1+\epsilon}$ which can be summarized in the following theorem:
Theorem A (see [OP, H, S]). Let $0 \le \epsilon < \infty$, and let μ be a positive measure on $\mathbb{D}$. Then there exists $\epsilon \ge 0$ such that

$$\left(\int_{\mathbb{D}} \sum_j |f_j(z)|^{1+2\epsilon} d\mu(z)\right)^{\frac{1}{1+2\epsilon}} \le (1+\epsilon) \sum_j \| f_j \|_{\mathcal{A}_{\epsilon-1}^{1+\epsilon}} \qquad (1)$$

if and only if $\mu(S(I)) \le (1+\epsilon)|I|^{1+2\epsilon}$, for some $\epsilon \ge 0$ and for all intervals $I \subset \mathbb{T}$.
Luecking (Th. 2.2 in [Lu]) showed that a similar characterization can be expressed in terms of pseudohyperbolic balls. Indeed, it is easy to show that when the exponent $\epsilon \ge 0$, the condition on the measure $\mu(S(I)) < (1+\epsilon)|I|^{1+\epsilon}$ for all intervals $I \subset \mathbb{T}$ is equivalent to the condition $\mu(B(z, 1+\epsilon)) < (1+\epsilon)^{1+2\epsilon}$, for all $z \in \mathbb{D}, 1+\epsilon = \frac{1}{2}\text{dist}(z, \partial\mathbb{D})$.
Theorem A was obtained by Oleinik and Pavlov [OP], and independently by Hastings [H] for $\epsilon = 1$ and Stegenga [S] when $\epsilon \ge 0$. Extensions of these results, where derivatives of the functions are considered on left inside of (1) can be found in [Lu].

Duren [D] extended Carleson's result to the range of exponents $\epsilon \geq 0$ obtaining a similar condition on the measure as the one given by Carleson in the case $\epsilon = 0$.

Theorem B (see [C, D]). Let $0 \leq \epsilon < \infty$ and let μ be a positive measure on $\mathbb{D}$. Then there exists $\epsilon \geq 0$ such that

$$\left(\int_{\mathbb{D}} \sum_j |f_j(z)|^{1+2\epsilon} d\mu(z)\right)^{\frac{1}{1+2\epsilon}} \leq (1+\epsilon) \sum_j \| f_j \|_{\mathcal{H}^{1+\epsilon}}$$

if and only if for some $\epsilon \geq 0, \mu(S(I)) \leq (1+\epsilon)|I|^{\frac{1+2\epsilon}{1+\epsilon}}$, for all intervals $I \subset \mathbb{T}$.

We will give a different proof of the sufficiency condition in Duren's result, which explicitly will show that it is enough to consider pseudohyperbolic balls instead of Carleson squares, although as we mentioned above they are equivalent since the exponent $\epsilon > 0$.

We have avoided up to this point the term Carleson measures because, the term is either used to define the measures for which the space $L^{1+\epsilon}(\mu)$ embeds continuously on the space of analytic functions under consideration or, it is used to define the geometric characterization of the measure in terms of Carleson squares. The theorems above show that, for the Hardy and for the weighted Bergman spaces in the disc, both definitions are actually equivalent, but it is not the case in a more general setting.

Given a Banach space X of analytic functions in a domain $\Omega \subset \mathbb{C}$ with norm $\|\cdot\|$, we say that a positive measure μ in Ω is a $(1+2\epsilon)$-Carleson measure for X if there exists a constant $\epsilon \geq 0$ such that

$$\left(\int_{\mathbb{D}} \sum_j |f_j(z)|^{1+2\epsilon} d\mu(z)\right)^{\frac{1}{1+2\epsilon}} \leq (1+\epsilon) \sum_j \| f_j \|_X$$

We are interested in characterizing the Carleson measures for spaces of analytic functions defined on bounded simply connected domains $\Omega \subset \mathbb{C}$. This problem has been initially studied by Zinsmeister in [Z] where he extended Carleson's result to more general domains, showing how the geometry of the domain plays a fundamental role.

To define the Hardy spaces $\mathcal{H}^{1+\epsilon}(\Omega)$ we need to assume initially that $\partial\Omega$ is locally rectifiable. Denoting by ds the arc length measure, we define for $0 \leq \epsilon < \infty$

$$\mathcal{H}^{1+\epsilon}(\Omega) = \left\{ f_j \text{ analytic in } \Omega; \int_{\partial\Omega} \sum_j |f_j(z)|^{1+\epsilon} ds < \infty \right\}$$

The theory of Hardy spaces is well understood when Ω is a chord-arc domain, that is a domain bounded by a chord arc curve. In this case, the functions in $\mathcal{H}^{1+\epsilon}(\Omega)$ can be characterized in terms of the nontangential maximal function and the area integral as in the classical case, (see [JK]). Recall that a locally rectifiable curve Γ is a chord-arc curve if $\ell_\Gamma(z_1, z_2) \leq (1+\epsilon)|z_1 - z_2|$ for some $\epsilon \geq 0$ and for all $z_1, z_2 \in \Gamma$, where $\ell_\Gamma(z_1, z_2)$ denotes the length of the shortest arc of Γ joining z_1 and z_2.

For Ω be a simply connected domain and $\varphi_j: \mathbb{D} \to \Omega$ a conformal maps. Let μ be a positive measure on Ω. When $\partial\Omega$ is rectifiable, a simple change of variables shows that μ being a $(1+2\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\Omega)$, that is,

$$\left(\int_{\Omega} \sum_j |f_j|^{1+2\epsilon} d\mu\right)^{\frac{1}{1+2\epsilon}} \leq (1+\epsilon) \left(\int_{\partial\Omega} \sum_j |f_j|^{1+\epsilon} ds\right)^{\frac{1}{1+\epsilon}}$$

is equivalent to

$$\left(\int_{\mathbb{D}} \sum_j |f_j \circ \varphi_j|^{1+2\epsilon} d\varphi_j^*(\mu)\right)^{\frac{1}{1+2\epsilon}} \leq (1+\epsilon) \left(\int_{\partial\mathbb{D}} \sum_j |f_j|^{1+\epsilon} |\varphi_j'| ds\right)^{\frac{1}{1+\epsilon}}$$

where $\varphi_j^*(\mu)$ denotes the pullback of μ, that is $\varphi_j^*(\mu)(E) = \mu(\varphi_j(E))$, for any set $E \subset \mathbb{D}$. Therefore, as in [Z], if we define the measure ν_j in $\mathbb{D}$ as $\nu_j = \frac{1}{|\varphi_j'|^{\frac{1+2\epsilon}{1+\epsilon}}} \varphi_j^*(\mu)$, we obtain the following observation that we state as a remark for further references.

Remark: The measure μ is a $(1+2\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\Omega)$ if and only if the measure ν_j is $(1+2\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\mathbb{D})$

We do not need to assume rectifiability on the boundary of the domain in order to study Carleson measures for $\mathcal{H}^{1+\epsilon}(\Omega)$

Theorem C ([Z]). Let Ω be a bounded simply connected domain and let $\varphi_j: \mathbb{D} \to \Omega$ be a conformal maps. Assume that $\log \varphi_j' \in BMOA(\mathbb{D})$. Then a positive measure μ in Ω is a $(1+\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\Omega), 0 \le \epsilon < \infty$, if for some $\epsilon \ge 0, \mu(B(\xi,(1+\epsilon)) \cap \Omega) \le (1+\epsilon)^2$, for all $\xi \in \partial\Omega$.

Moreover, it is also showed in [Z] that the condition on the domain: $\log \varphi_j' \in$ BMOA(D), is a necessary condition for Theorem C to hold.

The geometry of domains for which $\log \varphi_j' \in$ BMOA(D) has been characterized by Bishop and Jones in [BJ]. The boundary of these domains might not be rectifiable, and though we will not give the precise definition, we just mention that in some sense they are rectifiable most of the time on all scales. A typical example is a variant of the snowflake where at each iteration step, one of sides of the triangle, for instance the left one, is left unchanged.

To prove Theorem C in $[Z]$, it is first shown that the result holds for chord arc domains. The general result follows using the fact that when the BMO norm is small enough the domain is chord arc. We will give a different proof based on a stopping time argument.

For the converse result of Theorem C one needs a stronger assumption on the boundary of the domain. We say that a curve Γ is Ahlfors regular if there exists $\epsilon \ge 0$ such that for all $z_0 \in \Gamma$ and all $\epsilon \ge 0$, the arclength of $B(z_0,(1+\epsilon)) \cap \Gamma \le (1+\epsilon)^2$. These curves were studied by G. David in the context of the Cauchy integral [Da].

Chord arc curves are Ahlfors regular but not viceversa, for example cusps are Ahlfors regular but not chord arc. On the other hand, if we add the condition that the curve is a quasicircle, then Ahlfors regular and chord arc are equivalent.

Theorem D([Z]). Let Ω be a bounded simply connected domain, and μ be a positive measure in Ω. Assume that $\partial\Omega$ is an Ahlfors-regular Jordan curve. Then a positive measure μ in Ω is a $(1+\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\Omega), 0 \le \epsilon < \infty$, if and only if for some $\epsilon \ge 0, \mu(B(\xi,(1+\epsilon)) \cap \Omega) \le (1+\epsilon)^2$, for all $\xi \in \partial\Omega$.

The sufficiency condition follows immediately from Theorem C, since domains bounded by Ahlfors regular curves satisfy that $\log \varphi_j' \in \mathrm{BMOA}(\mathbb{D})$. The necessity is proved using Heyman-Wu Theorem. We will provide a simple proof in the case that $\partial\Omega$ is also a quasicircle, and therefore chord-arc.

We will state the next results in terms of Whitney balls in the domain Ω. They play the same role as pseudohyperbolic balls in the unit disc. We will say that a ball $B(z,1+\epsilon) \subset \Omega$ is a Whitney ball if $(1+\epsilon)\delta_\Omega(z) \le 1+\epsilon \le \frac{1}{2}\delta_\Omega(z)$, for some fixed constant $\epsilon \ge 0$ where $\delta_\Omega(z)$ denotes the distance from z to the boundary of Ω.

In Hardy spaces we obtain the analogous result of Duren's theorem in the classical case.

Theorem 1. Let Ω be a bounded simply connected domain,and μ a positive measure in Ω. If $0 \le \epsilon < \infty$, then μ is a $(1+\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\Omega)$ if and only if there is a constant $\epsilon \ge 0$ such that $\mu(B(z,1+\epsilon)) \le (1+\epsilon)^{\frac{1+3\epsilon}{1+\epsilon}}$, for all Whitney balls $B(z,1+\epsilon) \subset \Omega$.

Notice that in contrast with the classical case $\epsilon = 0$, no condition on the geometry of the domain is assumed. This is not surprising, it is a consequence that analogously as what happens in the disc, when the exponents are bigger than 1, the characterizations of the Carleson measures can be expressed in terms of Whitney balls instead of balls centered at the boundary.

We define the Bergman spaces $\mathcal{A}_{\epsilon-1}^{1+\epsilon}(\Omega)$ for $0 \le \epsilon < \infty$, as

$$\mathcal{A}_{\epsilon-1}^{1+\epsilon}(\Omega) = \left\{ f_j \text{ analytic in } \Omega; \int_\Omega \sum_j |f_j(z)|^{1+\epsilon} \delta_\Omega(z)^{\epsilon-1} dm(z) < \infty \right\}$$

We consider both analytic functions and quasi-subharmonic functions as well. For $u_j \ge 0$ be a locally bounded, measurable functions on Ω. We say that the functions u_j is $(1+\epsilon)$-quasi-nearly subharmonic if the following condition is satisfied:

$$u_j(a) \le \frac{1}{(1+\epsilon)} \int_{B(a,1+\epsilon)} \sum_j u_j \, dm \tag{2}$$

whenever $B(a, 1+\epsilon) \subset \Omega$.
One can view (2) as a weak mean value property. Besides of nonnegative subharmonic functions, it also holds for nonnegative powers of subharmonic functions, and for sub-solutions to a large family of second order elliptic equations, see ([P], [PR]).

Theorem 2. Let Ω be a bounded simply connected domain and μ be a positive measure on Ω. Let $0 \le \epsilon < \infty$. The following conditions are equivalent:
(i) There is $\epsilon \ge 0$ such that $\mu(B(z, 1+\epsilon)) \le (1+\epsilon)^{2+2\epsilon}$, for all Whitney balls $B(z, 1+\epsilon) \subset \Omega$.
(ii) There is $\epsilon \ge 0$ such that for any $(1+\epsilon)$-quasi-subharmonic function $g_j \ge 0$ in Ω

$$\left(\int_\Omega \sum_j g_j^{1+2\epsilon}(z) d\mu(z)\right)^{\frac{1}{1+2\epsilon}} \le (1+\epsilon)\left(\int_\Omega \sum_j g_j^{1+\epsilon}(z)\delta_\Omega(z)^{\epsilon-1} dm(z)\right)^{\frac{1}{1+\epsilon}}$$

(iii) The measure μ is a $(1+2\epsilon)$-Carleson measure for the space $\mathcal{A}_{\epsilon-1}^{1+\epsilon}(\Omega)$.
The proof of (i) implies (ii) is very similar to the one given by Luecking in [L] where he proves that the analogous statement holds for subharmonic functions in the unit disc.

1 Basic facts and definitions

The notation $A \simeq B$ means that there is a constant $(1+\epsilon)$ such that $1/(1+\epsilon).A \le B \le (1+\epsilon).A$ The notation $A \lesssim B (A \gtrsim B)$ means that there is a constant $1+\epsilon$ such that $A \le (1+\epsilon).B$ $(A \ge (1+\epsilon).B)$.
We denote by $\mathbb{T}$ the boundary of the unit disc, and $B(z_0, (1+\epsilon))$ the ball of radius $(1+\epsilon)$ centered at the point $z_0 \in \mathbb{C}$. If B is a ball, $2B$ is the ball with the same center as B and twice the radius of B, and similarly for squares. Given a domain $\Omega \in \mathbb{C}$, for any $z \in \Omega$ we set $\delta_\Omega(z) = \operatorname{dist}(z, \partial\Omega)$. If the context is clear we will drop the subindex Ω and simply write $\delta(z)$.
Given an interval $I \in \mathbb{T}$, and the corresponding Carleson square $S(I)$, we define the top of the square $T(S)$ as the set of points

$$T(S) = \left\{(1+\epsilon)e^{i\theta} : e^{i\theta} \in I, 1 - \frac{|I|}{2\pi} \le 1+\epsilon < 1 - \frac{|I|}{4\pi}\right\}$$

A locally integrable functions f_j belongs to the space BMO($\mathbb{T}$) if

$$\| f_j \|_* = \sup_I \frac{1}{|I|}\int_I \sum_j |f_j(x) - a_I| dx < \infty$$

where the supremum is taken over all arcs $I \in \mathbb{T}$ and $a_I = \frac{1}{|I|}\int_I f_j(y)dy$
It is a well known result, see for instance Th.1.2, Ch.VI in [G], that if $f_j \in \mathrm{BMO}(\mathbb{T})$, then

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{T}} \sum_j |f_j(\xi) - f_j(z)| P_z(\xi) |d\xi| = A < \infty$$

where $f_j(z) = \int_{\mathbb{T}} \sum_j P_z(\xi) f_j(\xi)|d\xi|$ is the Poisson integral of f_j. Moreover $A \simeq \| f_j \|_*$.
In particular, for any interval $I \in \mathbb{T}$, if z_I denotes the point $z_I = (1 - |I|/2)\xi_I$, being ξ_I the midpoint of the interval I, we have

$$\frac{1}{|I|}\int_I \sum_j |f_j(\xi) - f_j(z)| d\xi \le (1+\epsilon)\sum_j \| f_j \|_* \quad (3)$$

for some $(1+\epsilon) = (1+\epsilon)(\| f_j \|_*)$.
The space BMOA(D) consists of those functions in the Hardy space $\mathcal{H}^1(\mathbb{D})$ whose boundary values are in BMO($\mathbb{T}$). See [G] and [Po] for the main properties of BMOA functions.
For a harmonic functions f_j in $\mathbb{D}$ and for any $\xi \in \partial\mathbb{D}$, the nontangential maximal functions f_j^* is defined as $f_j^*(\xi) = \sup\{|f_j(z)| : z \in \Gamma_\xi\}$, where Γ_ξ denotes the cone $\Gamma_\xi = \{z \in \mathbb{D} : |z - \xi| < (1+\epsilon)(1 - |z|)\}$, for some fixed $\epsilon \ge 0$.
Given a functions $f_j \in L^1_{\mathrm{loc}}(\mathbb{R})$, the Hardy-Littlewood maximal function of f_j is

$$Mf_j(x) = \sup_{x\in I} \frac{1}{|I|}\int_I \sum_j |f_j(t)| dt$$

It is well known that the operator Mf_j is bounded in $L^{1+\epsilon}; 0 < \epsilon < \infty$ and it is of weak type $1-1$, that is

$$|t \in \mathbb{R} : Mf_j(t) > (1+\epsilon)| \le \frac{2}{(1+\epsilon)}\sum_j \| f_j \|_1, \epsilon > -1.$$

The importance of the maximal function Mf_j is that it majorizes many other functions associated with f_j, such as the non-tangential maximal function of the Poisson integral of f_j. Therefore, if $u_j(z) = P_z * f_j$, its non-tangential maximal function $u_j^*(x) = \sup_{\Gamma_x}|u_j(z)|$ satisfies

$$\left|x \in I; u_j^*(x) > (1+\epsilon)\right| \le \int_I \sum_j |f_j(x)|dx \tag{4}$$

for some constant $\epsilon \ge 0$ depending only on the aperture of the cone Γ_x, (see Chapter I.4 in [G]).
We mentioning some well-known results on conformal mappings, see Ch.I in [Po] for more knowledge.
If $\varphi_j: \Omega \to \mathbb{D}$ is conformal and $w = \varphi_j(z)$ then $\rho_\Omega(w_1, w_2) = \rho_{\mathbb{D}}(z_1, z_2)$ defines the hyperbolic metric on Ω and is independent of the particular choice of φ_j. It is often convenient to estimate ρ_Ω in terms of the more geometric quasi-hyperbolic metric on Ω which is defined as

$$\tilde{\rho}_\Omega(w_1, w_2) = \inf \int_{w_1}^{w_2} \frac{|dw|}{\delta_\Omega(w)}$$

where the infimum is taken over all arcs in Ω joining w_1 to w_2. It follows from Koebe $1/4$ theorem that the two metrics are comparable with bounds independent of the domain. A Whitney decomposition of the domain Ω is a covering of Ω by squares Q_k with disjoint interiors and the property that $\operatorname{diam}(Q_k) \simeq \delta_\Omega(Q_k)$. From the remarks above, each square in a Whitney decomposition has uniformly bounded hyperbolic diameter (and contains a ball with hyperbolic radius bounded uniformly from below). Thus bounding the hyperbolic length of a path often reduces to simply estimating the number of Whitney squares it hits.
Hence, we will say that a ball $B(z, 1+\epsilon) \in \Omega$ is a Whitney ball if $\delta_\Omega(z) \le 1+\epsilon \le 1/2\delta_\Omega(z)$, for some fixed constant $\epsilon \ge 0$.
We recall that the function $\log \varphi_j' \in \mathcal{B}$, where $\mathcal{B}$ denotes the Bloch space in $\mathbb{D}$. Therefore, for any Carleson square $S \in \mathbb{D}$, if z_1, z_2 are any two points in the top of the square $T(S)$, then

$$\left|\varphi_j'(z_1)\right| \simeq \left|\varphi_j'(z_2)\right|; z_1, z_2 \in T(S) \tag{5}$$

This is because the hyperbolic diameter of the top of the squares is uniformly bounded.

2 Proofs of some known results

We give a short proof of the characterizations of the $(1+2\epsilon)$-Carleson measures for $\mathcal{H}^{1+\epsilon}; \epsilon \ge 0$ due to Duren.

Proof. (Proof of Theorem B, $\epsilon \ge 0$) (see [Go])
It is enough to prove the result for that $\epsilon = 0$, so let $\epsilon > 0$. Assume first that for any Whitney ball $B(z, 1+\epsilon) \subset \mathbb{D}$

$$\mu\big(B(z, 1+\epsilon)\big) \le (1+\epsilon)^{2+\epsilon} \tag{6}$$

It is a well-known result that if $f_j \in \mathcal{H}^1$, then $f_j(z) \lesssim 1/(1-|z|)$ for $z \in \mathbb{D}$, and the non-tangential maximal functions $f_j^* \in \mathcal{H}^1$, see for example [G]. Using these results and Fubini's theorem, we get

$$\begin{aligned}\int_{\mathbb{D}} |f_j(z)|^{1+\epsilon} d\mu(z) &= \int_{\partial\mathbb{D}} \int_{\Gamma_\xi} \sum_j \frac{|f_j(z)|^{1+\epsilon}}{1-|z|} d\mu(z)|d\xi| \le \int_{\partial\mathbb{D}} \sum_j f_j^*(\xi) \int_{\Gamma_\xi} \frac{|f_j(z)|^{\epsilon}}{1-|z|} d\mu(z)|d\xi| \\ &\lesssim \int_{\partial\mathbb{D}} \sum_j f_j^*(\xi) \int_{\Gamma_\xi} \frac{1}{(1-|z|)^{1-\epsilon}} d\mu(z)|d\xi| \lesssim \int_{\partial\mathbb{D}} \sum_j f_j^*(\xi)|d\xi| \lesssim \sum_j \| f_j \|_{\mathcal{H}^1}\end{aligned}$$

since the integral on the cone Γ_ξ can be estimated in terms of Whitney balls B_k centered at points $z_k = (1-2^{-k})\xi; k = 0,1, \dots$ as

$$\int_{\Gamma_\xi} \frac{1}{(1-|z|)^{1-\epsilon}} d\mu(z) \lesssim \sum_k \mu(B_k)/(1-|z_k|)^{1-\epsilon} \lesssim \sum_k (2^{-k})^{2\epsilon} \lesssim 1$$

because μ satisfies (6) and $\epsilon > 0$
The converse result is standard, we give the idea of the proof for the sake of completeness. Assume that μ is a $(1+\epsilon)$-Carleson measure for $\mathcal{H}^1$. For each $z_0 \in \mathbb{D}$, choose $z_0^* \in \mathbb{C} \setminus \mathbb{D}$ such that $|z_0 - z_0^*| \sim 1 - |z_0| \sim 1 - |z_0^*|$. Then it is easy to prove that $f_j(z) = \frac{1}{(z-z_0^*)^2}$ is in $\mathcal{H}^1$ with $\| f_j \|_{\mathcal{H}^1} \sim 1/(1-|z_0|)$. The result now easily follows by applying the hypothesis to the functions f_j.

Next we prove Zinsmeister's result in Theorem C. Thus assuming that $\log \varphi_j' \in$ BMOA we want to show that, if $\mu(B(\xi, 1+\epsilon) \cap \Omega) \leq (1+\epsilon)^2; \xi \in \partial\Omega$ then μ is a Carleson measure for the Hardy spaces in Ω or, equivalently by the remark in the Introduction, that $\nu_j(B(\xi, 1+\epsilon) \cap \mathbb{D}) \leq (1+\epsilon)^2; \xi \in \partial\mathbb{D}, \epsilon \geq 0$.
To simplify the notation we will replace the ball $B(\xi, 1+\epsilon); \xi \in \partial\Omega$ by a Carleson square in the upper half plane, that is, a square with base on some interval $I \subset \mathbb{R}$. So, let $Q = \{(x,y); x \in I, 0 \leq y \leq |I|\}$. For any such square, we define the top of the square $T(Q) = \{(x,y); x \in I, 1/2 \leq y \leq |I|\}$, and the center of $T(Q)$ as the point $z_I = x_I + i3/2|I|$, where x_I denotes the midpoint of the interval I.

Proof. (Proof of Theorem C) (see [Go])

Let I be any interval in $[0,1]$, define the functions $(f_j)_I(z) = \log \varphi_j'(z) - \log \varphi_j'(z_I)$ and $(u_j)_I(z) = \operatorname{Re} f_j(z) = \log|\varphi_j'(z)| - \log|\varphi_j'(z_I)|$. Since $(f_j)_I \in \mathcal{H}^1$, the harmonic function $(u_j)_I$ is the Poisson integral of its boundary values, i.e. $(u_j)_I(z) = P_z * \left(\log|\varphi_j'(x)| - \log|\varphi_j'(z_I)|\right)$ (see Th.3.6, Ch.II in [G]).
On the other hand, by (3) we get

$$\frac{1}{|I|}\int_I \sum_j (u_j)_I(x)dx = \frac{1}{|I|}\int_I \sum_j \left|\log|\varphi_j'(x)| - \log|\varphi_j'(z_I)|\right| dx \lesssim \sum_j \| \log \varphi_j' \|_*$$

Therefore by (4), for any interval I

$$\frac{1}{|I|}\left|x \in I; (u_j)_I^*(x) > (1+\epsilon)\right| \lesssim \sum_j \frac{\|\log \varphi_j'(z)\|_*}{(1+\epsilon)} \tag{7}$$

Fix now an interval I and let Q be the corresponding Carleson square. We want to show that $\nu_j(Q) \leq (1+\epsilon)|I|$.
The idea is to divide Q into a countable union of disjoint regions, in such a way that $|\varphi_j'|$ behaves like a constant inside each of those regions. For that, we will use a stopping time argument.
For each $k = 1,2,\ldots$ form the 2^k intervals obtained by dividing I dyadically, and associate to each of these intervals the corresponding Carleson square. We obtained in this way 2^k squares of length $2^{-k}|I|$. Denote by $\{Q_{j_0}\}$ these collection of squares. Note that each Q_{j_0} is associated to some $k \in \mathbb{N}$ and it is contained in some other square Q_l associated to $k-1$, which we will call the father (if $k=1$, the father is Q). Squares with the same father will be called brothers.
Let $M_j > 0$ be a big enough constant that will be fixed later. Recall that z_I is the center of $T(Q)$. Define the first generation $G_1(Q)$ as those $Q_{j_0} \subset Q$ such that

$$\sup_{z \in T(Q_{j_0})} \sum_j \log|\varphi_j'(z)| - \sum_j \log|\varphi_j'(z_I)| > \log \sum_j M_j \tag{8}$$

and Q_{j_0} is maximal.
We also say that the interval $I_{j_0} \in G_1(I)$, if I_{j_0} is the base of some square $Q_{j_0} \in G_1(Q)$ Note that the intervals $\{I_{j_0}\}; I_{j_0} \in G_1(I)$ have pairwise disjoint interiors, and that in the region $\mathcal{R}_1(Q) = Q \setminus \cup_{G_1(Q)} Q_{j_0}$,

$$\sum_j \frac{1}{M_j}|\varphi_j'(z_I)| \leq \sum_j |\varphi_j'(z)| \leq \sum_j M_j \mid \varphi_j'(z_I); z \in \mathcal{R}_1 \tag{9}$$

On the other hand, if $x \in I_{j_0}$ for some $I_{j_0} \in G_1(I)$, its cone Γ_x contains points which belong to $T(Q_{j_0})$. So, by the choice of the squares in the first generation and by (5), we get that $(u_j)_I^*(x) > \log(1+\epsilon)M_j$, for some universal constant $\epsilon \geq 0$. Therefore, by (7)

$$\sum_{I_{j_0} \in G_1(I)} |I_{j_0}| \leq \left| x \in I; (u_j)_I^*(x) > (1+\epsilon)\sum_j M_j \right| \leq \frac{1}{10}|I| \tag{10}$$

if $M_j > (M_j)_0$ where $(M_j)_0$ is a constant depending only on $\|\log \varphi_j'(z)\|_*$.
Next, to each square in the first generation $Q_{j_0} \in G_1(Q)$ we apply the same stopping time rule (8), that is we start partitioning Q_{j_0} and we choose those maximal squares $Q_k \subset Q_{j_0}$ for which

$$\sup_{z \in T(Q_k)} \sum_j \left|\log|\varphi_j'(z)| - \log\left|\varphi_j'\left(z_{I_{j_0}}\right)\right|\right| > \log \sum_j M_j$$

where $z_{I_{j_0}}$ is now the center of $T(Q_{j_0})$. In this way we obtain a new generation of squares $G_2(Q_{j_0})$. Define the second generation as

$$G_2(Q) = \cup \left\{G_1\left(Q_{j_0}^1\right); Q_{j_0}^1 \subset G_1\right\} = \{Q_1^2, Q_2^2, \dots\}$$

Repeating the process with G_2 and continuing inductively we obtain later generations of squares $G_n(Q) = \{Q_1^n, Q_2^n, \dots\}$, and generations of intervals $G_n(I) = \{I_1^n, I_2^n, \dots\}, n \in \mathcal{N}$. Moreover, by (10),

$$\sum_{I_{j_0} \in G_n(I)} |I_{j_0}| \le \frac{1}{10} \sum_{I_{j_0} \in G_{n-1}(I)} |I_{j_0}| \le \cdots . \le \frac{1}{10^n}|I| \tag{11}$$

We are ready now to prove the theorem. Let Q be the initial Carleson square. By the definition of the measure ν_j

$$\nu_j(Q) = \int_Q \sum_j \frac{1}{|\varphi_j'(z)|} d\mu(z) = \sum_{n=1} \sum_{l;\mathcal{R}_l \in G_n} \int_{\mathcal{R}_l} \sum_j \frac{1}{|\varphi_j'(z)|} d\mu(z) \tag{12}$$

By construction, the region $\mathcal{R}_l \in G_n$ is contained in some square of the previous generation, that we will denote by Q_l. As it was observed in (9), if $z \in \mathcal{R}_l$ then $|\varphi_j'(z)| \simeq |\varphi_j'(z_l)|$, where z_l is the center of $T(Q_l)$.

Therefore $\sum_j |\varphi_j(z) - \varphi_j(w)| \simeq \sum_j |\varphi_j'(z_l)||z - w|$ for all $z, w \in \mathcal{R}_l$. Since the diameter of R_l is comparable to $\operatorname{Im} z_l$, Koebe's theorem implies that $\varphi_j(\mathcal{R}_l) \subset B\left(\varphi_j(z_l), \delta_\Omega(z_l)\right) \cap \Omega$. By the hypothesis on the measure μ

$$\mu\left(\varphi_j(\mathcal{R}_l)\right) \le \mu\left(B\left(\varphi_j(z_l), \delta_\Omega(z_l)\right) \cap \Omega\right) \lesssim \delta_\Omega(z_l) \simeq |\varphi_j'(z_l)| \operatorname{Im} z_l$$

We conclude by (12) and (11) that

$$\nu_j(Q) \simeq \sum_{n=1} \sum_{l;\mathcal{R}_l \in G_n} \sum_j 1/|\varphi_j'(z_l)| \mu\left(\varphi_j(\mathcal{R}_l)\right) \lesssim \sum_{n=1} \sum_{l;\mathcal{R}_l \in G_n} \operatorname{Im}(z_l) \lesssim \sum_{n=1} \sum_{j_0} |I_{j_0}^{n-1}| \lesssim \sum_{n=1} \frac{1}{10^n}|I|$$
$$\le (1+\epsilon)|I|$$

with comparison constants $(1+\epsilon) = (1+\epsilon)\left(\| \mu \|, \|\log \varphi_j'\|_*\right)$.

We give an alternative proof of the theorem by Zinsmeister, on the characterization of Carleson measures on domains bounded by Ahlfors regular curves, when we add the extra hypothesis that the curve is a quasicircle, and therefore chord arc.

Proof. (Proof of Theorem D for chord-arc domains) (see [Go])

Let Ω be a domain bounded by a chord arc curve Γ. Assume that μ satisfies a 1-Carleson measure for $\mathcal{H}^1(\Omega)$, that is, for all $f_j \in \mathcal{H}^1(\Omega)$,

$$\int_\Omega \sum_j |f_j| d\mu \lesssim \int_\Gamma \sum_j |f_j| ds \tag{13}$$

Let $B(\xi_0, (1+\epsilon))$ be a ball centered at $\xi_0 \in \partial\Omega$ of radius $\epsilon \ge 0$, and w_0 be a point in $B(\xi_0, (1+\epsilon)) \cap \Omega$ such that $\delta(w_0) \simeq (1+\epsilon)$. Since chord arc curves are quasicircles, by the circular distortion theorem they admit a quasiconformal reflection (see [A]). Thus we can choose a point $w_0^* \in \mathbb{C} \setminus \Omega$ such that for all $w \in B(\xi_0, (1+\epsilon)) \cap \Omega$, it holds that $|w - w_0^*| \sim \delta(w_0) \sim \delta(w_0^*)$.

Consider the function $f_j(z) = \frac{1}{(w-w_0^*)^2}$. It is easy to show that $f_j \in \mathcal{H}^1(\Omega)$ with $\| f_j \|_{\mathcal{H}^1} \lesssim 1/\delta(w_0)$. Indeed, consider the balls $B_k = B\left(w_0^*, 2^k\delta(w_0^*)\right); k = 1,2, \dots$, and the annuli $A_k = B_k \setminus B_{k-1}; k = 2,3, ..$ Then

$$\int_\Gamma \frac{1}{|w - w_0^*|^2} ds = \sum_{k=2} \int_{\Gamma \cap A_k} \frac{1}{|w - w_0^*|^2} ds \lesssim \sum_{k=2} \frac{1}{\left(2^k\delta(w_0^*)\right)^2} \operatorname{length}(\Gamma \cap B_k) \lesssim \frac{1}{\delta(w_0)}$$

because Γ is Ahlfors regular and therefore length $(\Gamma \cap B_k) \lesssim r(B_k) = 2^k\delta(w_0^*)$.

We can then use (13) to bound $\mu(B(\xi_0, (1+\epsilon)) \cap \Omega)$ as follows

$$\frac{\mu(B(\xi_0, (1+\epsilon)) \cap \Omega)}{\delta(w_0)^2} \simeq \int_{B(\xi_0,(1+\epsilon))\cap\Omega} \frac{1}{|w - w_0^*|^2} d\mu(z) \lesssim \int_\Gamma \frac{1}{|w - w_0^*|^2} ds \lesssim \frac{1}{\delta(w_0)}$$

Hence, $\mu(B(\xi_0, (1+\epsilon)) \cap \Omega) \lesssim \delta(w_0) \simeq (1+\epsilon)$ as we wanted to show.

3 Proofs of Theorems 1 and 2

Proof. (Proof of Theorem 1) (see [Go])

Let φ_j represent a conformal mapping from $\mathbb{D}$ onto Ω. Assume first that the measure μ satisfies that

$$\mu\left(B(w, 1+\epsilon)\right) \le (1+\epsilon)^{\frac{2+3\epsilon}{1+\epsilon}} \tag{14}$$

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for all Whitney balls $B(w, 1+\epsilon) \subset \Omega$.

Define the measure ν_j in $\mathbb{D}$ as $d\nu_j = \frac{1}{|\varphi_j'|^{\frac{1+2\epsilon}{1+\epsilon}}} d\varphi_j^*(\mu)$. By the remark in the Introduction, showing that μ is a Carleson measure in Ω is equivalent to showing that ν_j is a Carleson measure in $\mathbb{D}$. So, let $B(z_0, 1+\epsilon) \subset \mathbb{D}$ be a Whitney ball in $\mathbb{D}$. As $|\varphi_j'(z)| \simeq |\varphi_j'(z_0)|$ when $z \in B(z_0, 1+\epsilon)$, we get

$$\nu_j\big(B(z_0, 1+\epsilon)\big) = \int_{B(z_0,1+\epsilon)} \sum_j \frac{1}{|\varphi_j'(z)|^{\frac{1+2\epsilon}{1+\epsilon}}} d\varphi_j^*(\mu(z)) \simeq \sum_j \frac{1}{|\varphi_j'(z_0)|^{\frac{1+2\epsilon}{1+\epsilon}}} \int_{\varphi_j(B(z_0,1+\epsilon))} d\mu$$

Set $w_0 = \varphi_j(z_0)$. As explained in section 1, $\varphi_j\big(B(z_0, 1+\epsilon)\big)$ can be covered by a finite union of Whitney balls $B\big(w_{j_0}, r_{j_0}\big) \subset \Omega; j_0 = 1 \dots, N_0$ where N_0 is a universal constant and $r_{j_0} \simeq \delta\big(w_{j_0}\big) \simeq \delta(w_0)$. Therefore, by (14) and by Koebe's theorem

$$\nu_j\big(B(z_0, 1+\epsilon)\big) \simeq \sum_j \frac{1}{|\varphi_j'(z_0)|^{\frac{1+2\epsilon}{1+\epsilon}}} \mu\Big(\varphi_j\big(B(w_0, (1+\epsilon))\big)\Big) \lesssim \sum_j \frac{1}{|\varphi_j'(z_0)|^{\frac{1+2\epsilon}{1+\epsilon}}} \sum_{j_0} \mu\Big(B\big(w_{j_0}, r_{j_0}\big)\Big)$$

$$\leq \sum_j \frac{1}{|\varphi_j'(z_0)|^{\frac{1+2\epsilon}{1+\epsilon}}} \sum_{j_0} r_{j_0}^{\frac{1+2\epsilon}{1+\epsilon}} \simeq \sum_j \frac{1}{|\varphi_j'(z_0)|^{\frac{1+2\epsilon}{1+\epsilon}}} \delta(w_0)^{\frac{1+2\epsilon}{1+\epsilon}} \simeq (1-|\,z_0)^{\frac{1+2\epsilon}{1+\epsilon}} \simeq (1+\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}}$$

which implies by Theorem B that ν_j is a Carleson measure in $\mathbb{D}$.

To prove the "only if" part we proceed in a similar way. Assume now that μ is a $(1+2\epsilon)$-Carleson measure for $\mathcal{H}^{1+\epsilon}(\Omega)$. This is equivalent, by the remark in the Introduction and Duren's result in Theorem B, to the assumption that $\nu_j(B(z, 1+\epsilon)) \lesssim 1+\epsilon$ for all Whitney balls $B(z, 1+\epsilon) \subset \mathbb{D}$.

Next, consider a Whitney ball $B(w_0, (1+\epsilon)) \subset \Omega$, i.e. $(1+\epsilon) \sim \delta(w_0)$, and let $\varphi_j^{-1}(w_0) = z_0$.

The set $\varphi_j^{-1}\big(B(w_0, (1+\epsilon))\big)$ is contained in a finite union of Whitney balls $B\big(z_{j_0}, r_{j_0}\big) \subset \mathbb{D}; j_0 = 1..., J_0$ for some universal constant J_0. Thus, $|\varphi_j'(z)| \simeq |\varphi_j'(z_0)|$ when $z \in \cup_{j_0} B\big(z_{j_0}, r_{j_0}\big)$, and we obtain that

$$\mu(B(w_0, (1+\epsilon)) = \int_{B(w_0,(1+\epsilon))} d\mu = \int_{\varphi_j^{-1}(B(w_0,(1+\epsilon)))} \sum_j |\varphi_j'|^{\frac{1+2\epsilon}{1+\epsilon}} d\nu_j \leq \sum_{j_0} \int_{B(z_{j_0}, r_{j_0})} \sum_j |\varphi_j'|^{\frac{1+2\epsilon}{1+\epsilon}} d\nu_j$$

$$\lesssim \sum_{j_0} \sum_j |\varphi_j'(z_{j_0})|^{\frac{1+2\epsilon}{1+\epsilon}} \nu_j\Big(B\big(z_{j_0}, r_{j_0}\big)\Big) \lesssim \sum_{j_0} \sum_j |\varphi_j'(z_0)|^{\frac{1+2\epsilon}{1+\epsilon}} \big(1-|z_{j_0}|\big)^{\frac{1+2\epsilon}{1+\epsilon}} \simeq (1+\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}}$$

since $1-|z_{j_0}| \simeq 1-|z_0|$, and by Koebe's theorem $\delta(w_0) \simeq \varphi_j'(z_0)(1-|z_0|)$.

Proof. (Proof of Theorem 2) (see [Go])

$(i) \Rightarrow (ii)$ To simplify notation set $\delta(z) = \delta_\Omega(z)$. Let us begin with a simple observation related to Whitney balls: If $w \in B\left(z, \frac{1}{4}\delta(z)\right)$, then $z \in B\left(w, \frac{1}{3}\delta(w)\right)$. Besides $\delta(w) \simeq \delta(z)$.

To see this, just note that $\delta(z) \leq |z-w| + \delta(w)$. Therefore, if $|w-z| \leq \frac{1}{4}\delta(z)$, then $|w-z| \leq \frac{1}{4}(|z-w| + \delta(w))$, that is, $|w-z| \leq \frac{1}{3}\delta(w)$. The rest of the statement follows by a similar argument.

Let g_j be a C-quasi-subharmonic function, then for any $\epsilon \geq 0$, $g_j^{1+\epsilon}$ is K-quasisubharmonic with constant $K = K(C, 1+\epsilon)$[P]. Therefore, for all $z \in \Omega$ and any Whitney ball $B(z, 1+\epsilon)$

$$g_j^{1+\epsilon}(z) \lesssim \frac{1}{(1+\epsilon)^2} \int_{B(z,1+\epsilon)} g_j^{1+\epsilon}(w)\, dm(w)$$

We can then write

$$\left(\int_\Omega \sum_j g_j^{1+2\epsilon}(z)\, d\mu(z)\right)^{\frac{1+\epsilon}{1+2\epsilon}} \lesssim \left(\int_\Omega \left(\frac{1}{\delta(z)^2} \int_{B\left(z, \frac{1}{4}\delta(z)\right)} \sum_j g_j^{1+\epsilon}(w)\, dm(w)\right)^{\frac{1+2\epsilon}{1+\epsilon}} d\mu(z)\right)^{\frac{1+\epsilon}{1+2\epsilon}} \quad (15)$$

By the observation above, the integral in (15) can be bounded by

$$\left(\int_{\Omega}\left(\int_{\Omega}\sum_{j}\frac{1}{\delta(w)^2}\chi_{B\left(w,\frac{1}{3}\delta(w)\right)}(z)g_j^{1+\epsilon}(w)dm(w)\right)^{\frac{1+2\epsilon}{1+\epsilon}}d\mu(z)\right)^{\frac{1+\epsilon}{1+2\epsilon}}$$

Since $\epsilon \geq 0$, by Minkowski's integral inequality and the condition (i) on the measure, we obtain

$$\left(\int_{\Omega}\sum_{j} g_j^{1+2\epsilon}(z)d\mu(z)\right)^{\frac{1+\epsilon}{1+2\epsilon}} \lesssim \int_{\Omega}\left(\int_{\Omega}\sum_{j}\left(\frac{1}{\delta(w)^2}\chi_{B\left(w,\frac{1}{3}\delta(w)\right)}(z)g_j^{1+\epsilon}(w)\right)^{\frac{1+2\epsilon}{1+\epsilon}}d\mu(z)\right)^{\frac{1+\epsilon}{1+2\epsilon}}dm(w)$$

$$\lesssim \int_{\Omega}\sum_{j}\frac{1}{\delta(w)^2}g_j^{1+\epsilon}(w)(\mu(B)(w,\frac{1}{3}\delta(w)))^{\frac{1+\epsilon}{1+2\epsilon}}dm(w) \lesssim \int_{\Omega}\sum_{j} g_j^{1+\epsilon}(w)\delta(w)^{\epsilon-1}dm(w)$$

with comparison constants only depending on $((1+\epsilon),1+\epsilon,\|\mu\|)$ as we wanted to prove.

$(ii) \Rightarrow (iii)$ This is an immediate consequence of the fact that if f_j is analytic, then $|f_j|$ is subharmonic.

$(iii) \Rightarrow (i)$ Let φ_j represent a conformal mapping from $\mathbb{D}$ onto Ω. By Koebe's theorem and a change of variables, condition (iiii) can be written as

$$\left(\int_{\mathbb{D}}\sum_{j}|f_j\circ\varphi_j|^{1+2\epsilon}d\varphi_j^*(\mu)\right)^{\frac{1}{1+2\epsilon}} \leq (1+\epsilon)\left(\int_{\mathbb{D}}\sum_{j}|f_j\circ\varphi_j|^{1+\epsilon}\left|\varphi_j'\right|^{1+\epsilon}(1-|z|)^{\epsilon-1}dm\right)^{\frac{1}{1+\epsilon}} \quad (16)$$

Defining the measure τ in $\mathbb{D}$ as $d\tau = \frac{1}{\left|\varphi_j'\right|^{1+2\epsilon}}d\varphi_j^*(\mu)$, (16) is equivalent to

$$\left(\int_{\mathbb{D}}\sum_{j}\left(|f_j\circ\varphi_j|\left|\varphi_j'\right|\right)^{1+2\epsilon}d\tau\right)^{\frac{1}{1+2\epsilon}} \leq (1+\epsilon)\left(\int_{\mathbb{D}}\sum_{j}\left(|f_j\circ\varphi_j|\left|\varphi_j'\right|\right)^{1+\epsilon}(1-|z|)^{\epsilon-1}dm\right)^{\frac{1}{1+\epsilon}}$$

which implies by Theorem A, that τ is a $(1+2\epsilon)$-Carleson measure in $\mathbb{D}$ for $\mathcal{A}_{\epsilon-1}^{1+\epsilon}$.

Consider now a Whitney ball $B(w_0,(1+\epsilon)) \subset \Omega$. To show that $\mu(B(w_0,(1+\epsilon)) \lesssim (1+\epsilon)^{1+2\epsilon}$ we follow exactly the same steps as in "only if part in Theorem 1, just replacing the exponent $\frac{1+2\epsilon}{1+\epsilon}$ by $1+2\epsilon$ and using the fact that since τ is $(1+2\epsilon)$-Carleson measure in $\mathbb{D}$ for $\mathcal{A}_{\epsilon-1}^{1+\epsilon}$, by theorem A, $\tau(B(z,(1+\epsilon)) \lesssim (1+\epsilon)^{1+2\epsilon}$ for all Whitney balls $B(z,(1+\epsilon)) \subset \mathbb{D}$.

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