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Approximation through Bayes of Scale In Weibull Model With Linex Loss Function

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ABSTRACT: *In life testing problems engineers must often deal with lifetimes data that are non-homogeneous. The two-component Weibull mixture is then a highly relevant model to capture heterogeneity for a large majority of operating lifetimes. Unfortunately, the performance of classical estimation methods is risked due to the high number of parameters. The Weibull mixture parameters estimation, in this research we propose a Bayesian Approximation by Lindley approach to provide the posterior density. In this paper we dealt with the estimation of an unknown scale parameter of the two parameter Weibull distribution with LINEX loss function suggested in this paper. It deals with the methods to obtain the approximate Bayes estimators of the Weibull distribution by using Lindley approximation technique for type-II censored samples. A bivariate prior density for the parameters, LINEX loss function, are used to obtain the approximate Bayes Estimators. A numerical calculation is done for approximate Bayes estimator and its relative mean squared errors by R programming to present the statistical properties of the estimators.*
Keywords: *Weibull Density, Lindley Approximation, Bayesian Technique, LINEX Loss, Censoring.*

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I. INTRODUCTION

Weibull distribution has been extensively used in life testing and reliability probability problems. The distribution is named after the Swedish scientist Weibull who proposed if for the first time in 1939 in connection with his studies on strength of material. Weibull (1951) showed that the distribution is also useful in describing the wear out of fatigue failures. Estimation and properties of the Weibull distribution is studied by many authors see Kao (1959).

The probability density function, reliability and hazard rate functions of Weibull distribution is given respectively as

$$f(x) = p\theta x^{(p-1)} \exp(-\theta\, x^P) \quad ; \quad x, \theta, p > 0 \tag{1.1}$$

$$R(t) = exp(-\theta t^p) \quad ; \quad t > 0 \tag{1.2}$$

$$H(t) = p\theta t^{(p-1)} \quad ; \quad t > 0 \tag{1.3}$$

Where $'\theta'$ is the scale and 'p' is shape parameters.

Ferguson (1985). Canfield (1970), Basu and Ebrahimi (1991). Zellner (1986) asymmetric loss function for a number of distributions. Such as a loss function is derived as

$$L(\Delta) = b\exp(a\Delta) - c\Delta - b; \quad \Delta = (\hat{\theta} - \theta), \tag{1.5}$$

and $a, c \neq 0$, $b > 0$

For a minimum to exist at Δ=0.

$$\left[\frac{\partial}{\partial\Delta} L(\Delta)\right]_{\Delta=0} = 0 = ab - c$$

And we have a two-parameter loss function

$$L(\Delta) = b\,[\exp(a\Delta) - a\Delta - 1];\ b>0, a>0, \tag{1.6}$$

The sign and magnitude of 'a' represents the direction and degree of symmetry respectively, when a>0, then overestimation is more serious than underestimation and vice-versa. For 'a' closed to zero the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of LINEX loss function in equation (1.5) is

$$E_\pi\left(L\left(\hat{\theta}-\theta\right)\right) \propto e^{a\hat{\theta}}E_\pi\left(e^{-a\theta}\right)-a\left(\hat{\theta}-E_\pi(\theta)\right)-1 \quad (1.7)$$

where E_π is the posterior expectation with respect to posterior density of θ.
The Bayes Estimator $\hat{\theta}_{BL}$ of θ under LINEX loss function is the value which minimizes equation (6) is

$$\hat{\theta}_{BL} = = -\frac{1}{a}log\left(E_\pi\left(e^{-a\theta}\right)\right) \quad (1.8)$$

provided that $E_\pi\left(e^{-a\theta}\right)$ exists and is finite. [Calabria and Pulcini (1996)].

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data in to the prior distribution using the Bayes theorem, the first theorem of inference, hence we update the prior distribution in the light of observed data. Thus the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution.

Linex Loss function is used to obtain the approximate Bayes Estimate for scale parameter of the Weibull distribution. A statistical software R is used for numerical calculations for different approximate Bayes estimators and their relative mean squared errors by preparing programs to present the statistical properties of the estimators

II. The Estimators:

Let $x_1, x_2, \dots\dots\dots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (1.1) The failure times are recorded as they occur until a fixed number 'r' of times failed. Let $= \left(x_{(1)}, x_{(2)}, \dots\dots\dots\dots\dots, x_{(n)}\right)$, where $x_{(i)}$ is the life time of the ith item. Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.
The likelihood function can be written as

$$L(x|\theta,p) = \frac{n!}{(n-r)!}(\mathrm{p}\theta)^r \prod_{i=1}^{r} x_i^{(p-1)} \exp(-\delta\theta), \quad (2.1)$$

Where:

$$\delta = \sum_{i=1}^{r} x_i^p + (n-r)x_r^p$$

The logarithm of the likelihood function is

$$\log L(x|\theta,p) \propto r\,log\,p + r\log\theta + (p-1)\sum_{i=1}^{r}\log x_i - \delta\theta, \quad (2.2)$$

assuming that 'p' is known, the maximum likelihood estimator $\hat{\theta}_{ML}$of θ can be obtain by using equation (2.2) as

$$\hat{\theta}_{ML} = r/\delta \quad (2.3)$$

If both the parameters p and θ are unknown their MLE's $\hat{p}_{ML}$and $\hat{\theta}_{ML}$ can be obtained by solving the following equation

$$\frac{\delta}{\delta\theta}\log L = \frac{r}{\theta} - \delta = 0 \ , \quad (2.4a)$$

$$\frac{\delta \log L}{\delta P} = \frac{r}{P} + \sum_{i=1}^{r}\log x_i - \theta\delta_1 = 0, \quad (2.4b)$$

Where

$\delta_1 = \sum_{i=1}^{r} x_i^P \log x_i + (n-r)x_r^P \log x_r$, eliminating θ between the two equations of (2.4) and simplifying we get

$$\hat{p}_{ML} = \frac{r}{\delta^*} \quad (2.5)$$

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} logx_i\right]$

Equation (2.5) may be solved for Newton- Raphson or any suitable iterative Method and this value is substituted in equation (2.4b) by replacing with p get $\hat{p}$ as

$$\hat{\theta}_{ML} = \frac{\frac{r}{\hat{p}_{ML}} + \sum_{i=1}^{r} logx_i}{\sum_{i=1}^{r} x_i^{\hat{p}_{ML}}\ logx_i + (\mathrm{n-r})x_r^{\hat{p}_{ML}}\ logx_r} \quad (2.6)$$

The MLE's of R(t) and H(t) are given respectively by equation (1.2) and (1.3) after replacing θ and p by $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

III. Bayes Estimator of θ when shape Parameter P is known:

If p is known assume gamma prior $\gamma(\alpha,\beta)$ as cojugate prior for θ as

$$g\left(\theta|\underline{x}\right) = \frac{\beta^\alpha}{\Gamma\alpha}(\theta)^{(\alpha+1)}\exp(-\beta\theta)\,;(\alpha,\beta) > 0, \theta > 0 \ , \quad (3.1)$$

The posterior distribution of θ using equation (2.1) and (3.1) we get

$h(\theta|\underline{x}) = \frac{(\delta+\beta)^{r+\alpha}}{\Gamma(r+\alpha)}(\theta)^{(r+\alpha-1)}\exp(-\theta(\delta+\beta))$, (3.2)

Under linex Loss Function, the Bayes estimator $\hat{\theta}_{BE}$ of θ using (1.7) and (3.2) given by

$\hat{\theta}_{BL} = \frac{(r+\alpha)}{a}\ log\left[1+\frac{a}{(\delta+\beta)}\right]$ (3.3)

IV. The Bayes estimators with θ and p unknown:

The joint prior density of θ and p is given by

$G(\theta|p) = g_1(\theta|p).g_2(p)$

$G(\theta|p) = \frac{1}{\lambda\Gamma\xi}p^{-\xi}\theta^{(\xi-1)}.\exp\left[-\left(\frac{\theta}{p}+\frac{p}{\lambda}\right)\right]$; $(\theta, p, \lambda, \xi) > 0$, (4.1)

where ;

$g_1(\theta|p) = \frac{1}{\Gamma\xi}p^{-\xi}\ \theta^{(\xi-1)}.\exp\left[-\frac{\theta}{p}\right]$ (4.2)

And

$g_2(p) = \frac{1}{\lambda}\exp\left(-\frac{p}{\lambda}\right)$; (4.3)

The joint posterior density of θ and p is

$h^*(\theta, p|\underline{x}) = \frac{\frac{1}{\lambda\Gamma\xi}p^{-\lambda}\theta^{(\xi+1)}\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right](\mathrm{p}\theta)^r\prod_{i=1}^{r}x_i^{(p-1)}e^{-p\theta}}{\iint\frac{1}{\lambda\Gamma\xi}p^{(r-\xi)}\theta^{(r+\xi+1)}\prod_{i=1}^{r}x_i^{(p-1)}.\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}+\mathrm{p}\theta\right\}\right]d\theta dp}$ (4.4)

V. Approximate Bayes Estimators under Linex loss function:

The Bayes estimators of a function $\mu = \mu(\theta, p)$ of the unknown parameter θ and p under Linex loss function is the posterior mean

$\hat{\mu}_{ABS} = E(\mu|\underline{x}) = \frac{\iint\mu(\theta,p)G(\theta,p|\underline{x})d\theta dp}{\iint G(\theta,p|\underline{x}).d\theta.dp}$; (5.1)

To evaluate (5.1) consider the method of Lindley approximation

$E(\mu(\theta,p)|\underline{x}) = \frac{\int\mu(\theta).e^{(l(\theta)+\rho(\theta))}d\theta}{\int e^{(l(\theta)+\rho(\theta))}d\theta}$; (5.2)

Where $(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function

The Lindley approximation for two parameter is given by

$E(\hat{\mu}(\theta,p)|\underline{x}) = \mu(\theta,p)+\frac{A}{2}+\rho_1A_{12}+\rho_2A_{21}+\frac{1}{2}[l_{30}B_{12}+l_{21}C_{12}+l_{12}C_{21}+ \quad l_{03}B_{21}]$

(5.3)

Where

$A = \sum_1^2\sum_1^2\mu_{ij}\sigma_{ij}$; $\quad , l_{\eta\epsilon} = (\delta^{(\eta+\epsilon)}l|\delta\theta_1^{\eta}\delta\theta_2^{\epsilon})$; (5.4)

Where$(\eta+\epsilon) = 3$ for $i,j = 1{,}2$ $\quad \rho_i = (\delta\rho|\delta\theta_i)$; (5.5)

$\mu_i = \frac{\delta\mu}{\delta\theta_i}$; , $\quad \mu_{ij} = \frac{\delta^2\mu}{\delta\theta_i\delta\theta_j}$; $\forall i \neq j$; (5.6)

$A_{ij} = \mu_i\sigma_{ij} + \mu_j\sigma_{ji}$; (5.7)

$B_{ij} = (\mu_i\sigma_{ii}+\mu_j\sigma_{ij})\sigma_{ii}$ (5.8)

$c_{ij} = 3\mu_i\sigma_{ii}\sigma_{ij}+\mu_j(\sigma_{ii}\sigma_{jj}+2\sigma_{ij}^2)$; (5.9)

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1{,}2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\theta_i\delta\theta_j}$.

All the function in (5.4-5.9) are evaluated at MLE of (θ_1,θ_2).In our case $(\theta_1,\theta_2) = (\theta,p); So\ \mu(\theta) = \mu(\theta,p)$

To apply Lindley approximation (5.2), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1{,}2$,which can be shown to be

$\sigma_{11} = \frac{M}{D}$, $\sigma_{12} = \sigma_{21} = \frac{\delta_1}{D}$,$\sigma_{22} = \frac{r}{D\,\theta^2}$, (5.10)

Where$M = \left(\frac{r}{p^2}+\theta\delta_2\right)$; $D = \left[\frac{r}{\theta^2}\left(\frac{r}{p^2}+\theta^2\delta_2\right)\right]$; (5.11)

$\delta_2 = \sum_{i=1}^{r}x_i^p(logx_i)^2+(n-r)x_r^p(logx_r)^2$; (5.12)

To evaluate ρ_i , take the joint prior $G(\theta|p)$

$G(\theta|p) = \frac{1}{\lambda\Gamma\xi}p^{-\xi}\theta^{(\xi-1)}.\exp\left[\left\{-\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right]$; $(\theta,p,\lambda,\xi) > 0$, (5.13)

$$\Rightarrow \rho = \log[G(\theta|p)] = constant - \xi logp - (\xi-1)log\theta - \frac{\theta}{p} - \frac{p}{\lambda}$$

Therefore

$\rho_1 = \frac{\partial\rho}{\partial\theta} = \frac{(\xi-1)\theta}{\theta} - \frac{1}{p}$; (5.14)

and

$\rho_2 = \frac{\theta}{p^2} - \frac{1}{\lambda} - \frac{\xi}{p};$ (5.15)

Further more

$l_{21} = 0\,; l_{12} = -\delta_2\,; l_{03} = \frac{2r}{p^3} - \theta\delta_3;$ (5.16)

$and\ \ l_{30} = \frac{2r}{\theta^3};$ (5.17)

Where $\delta_3 = \sum_{i=1}^{r} x_i^p\,(log x_i)^3 + (n-r)x_r^p(log x_r)^3$

$+\frac{rM^2}{\theta} - \frac{rM\delta_1}{2} - \theta^2\delta_1^2\delta_2 + \frac{r^2}{p^3}\delta_1 - \frac{\theta r\delta_1\delta_3}{2}];$

The Approximate Bayes estimator of a function $\mu = \mu(\theta, p)$ of unknown scale parameter θ under LINEX loss function in equation (1.6) is given by

$\hat{\mu}_{ABL} = -\frac{1}{a}log\left(E_{h^*}(e^{-a\mu})\right)\,;$ (5.18)

where

$E_{h^*}\left((e^{-a\mu}\,)\middle|\bar{x}\right) = \frac{\iint e^{-a\mu}\,h^*(\theta,p)d\theta\,dp}{\iint\,h^*(\theta,p)d\theta\,dp};$

we apply Lindley's Procedure to obtain Approximate Bayes estimators for scale parameter θ under Linex loss function (LLF) as

Approximate Bayes estimator of θ under Linex loss function (LLF) is

$\hat{\theta}_{ABL} = \theta + -\frac{1}{a}log\left[\frac{Ma^2}{2\,D} - a\,Q_1\right];$ at$\left(\hat{\theta}_{ML}, \hat{p}_{ML}\right)$, (5.19)

Where;

$Q_1 = \frac{1}{\theta^2 D^2}[\frac{M\theta D}{p}(p(\xi-1)-1) + \frac{\theta^2\delta_1 D}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\}$

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