International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 12 Issue 12 ǁ December 2024 ǁ PP. 245-252

Solutions of Couples of linear partial differential equations and nonlinear partial differential equations by using Variational Iteration Method

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Abstract
In this paper, we introduce the solution of systems of linear PDEs and nonlinear PDEs subject to the general initial conditions by using THE variational iteration method. Six illustrated examples has been introduced. The steps of the method are easy implemented and high accuracy.

Keywords: *System of linear partial differential equations, system of nonlinear partial differential equations, variational iteration method.*

Date of Submission: 11-12-2024 Date of acceptance: 23-12-2024

I. INTRODUCTION

Systems of partial differential equations, linear or nonlinear, have attracted much concern in studying evolution equations that describe wave propagation, in investigating shallow water waves, and in examining the chemical reaction-diffusion model of Brusselator.

The Variational Iteration Method (VIM) was first developed by Chinese mathematician Ji-Huan He, professor at Donghua University. The VIM was initially proposed toward the end of the most recent century and completely grew in 2006 and 2007.

The method VIM is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations, which converge rapidly to the accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

The variational iteration method (VIM) is relatively new approaches to provide approximate solutions to linear and nonlinear problems. The variational iteration method, (VIM) was successfully applied to find the solutions of several classes of variational problems.

Here we used VIM for solving systems of linear or nonlinear PDEs with initial conditions. This paper is arranged as follows. In section 2, Variational Iteration Method (VIM) 'S form. In section 3 Variational Iteration Algorithm for PDEs. In section 4, numerical examples. The conclusions in section 5.

II. VIM 'S FORM :

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)[u_n'(\xi) + \cdots + f(u_n, u_n', \dots)]d\xi$$

Firstly we find $\lambda(\xi)$: λ is a general Lagrange multiplier:

If $\mathcal{L} = \frac{d}{dx} \Rightarrow \lambda(\xi) = (-1)^1(1) = -1$

If $\mathcal{L} = \frac{d^2}{dx^2} \Rightarrow \lambda(\xi) = (-1)^2(\xi - x) = \xi - x$

If $\mathcal{L} = \frac{d^3}{dx^3} \Rightarrow \lambda(\xi) = \frac{(-1)^3(\xi - x)^2}{2!}$

⋮

Etc

secondly we use initial condition to chose $u_0(x)$

$$u_0(x) = \begin{cases} u(0) & for\ L = \dfrac{d}{dx} \\ u(0) + xu'(0) & for\ L = \dfrac{d^2}{dx^2} \\ u(0) + xu'(0) + \dfrac{1}{2!}x^2u'' & for\ L = \dfrac{d^3}{dx^3} \end{cases}$$

and so on .

III. VARIATIONAL ITERATION ALGORITHM for PARTIAL DIFFERENTIAL EQUATIONS:

Consider the following partial differential equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - u^3(x,y) = 0.$$

Now we begin with

$$u_0(x,y) = u(0,0) + xu_x(0,y) + yu_y(0,y) + yu_y(x,0).$$

Then the variational iteration algorithm can obtained as follows

$$\begin{cases} u_{n+1}(x,y) = u_n(x,y) + \displaystyle\int_0^x (\xi - x)\left\{\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi,y)}{\partial y^2} - u_n^3(\xi,y)\right\} d\xi, \\ u_{n+1}(x,y) = u_0(x,y) + \displaystyle\int_0^x (\xi - x)\left\{\frac{\partial^2 u_n(\xi,y)}{\partial y^2} - u_n^3(\xi,y)\right\} d\xi, \\ u_{n+1}(x,y) = u_n(x,y) + \displaystyle\int_0^x (\xi - x)\left\{\frac{\partial^2 u_n(\xi,y)}{\partial y^2} - u_n^3(\xi,y) - \frac{\partial^2 u_{n-1}(\xi,y)}{\partial y^2} + u_{n-1}^3(\xi,y)\right\} d\xi, \end{cases}$$

IV. NUMERICAL EXAMPLES :

In this section , we apply the variational iteration method (VIM) to solve systems of partial differential equations . Numerical results are very encouraging.

Example (1). Consider the following linear system:

$u_t + v_x - (u + v) = 0,$

$v_t + u_x - (u + v) = 0,$

$u(x,0) = \sinh x\, , v(x,0) = \cosh x,$

Solution :

Let's consider the linear equations:

$u_t + v_x - (u + v) = 0,\ \ u(x,0) = \sinh x$

$v_t + u_x - (u+v) = 0, \quad v(x,0) = \cosh x$

Then the correction functional is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} - u_n(x,\xi) - v_n(x,\xi) \right) d\xi,$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - u_n(x,\xi) - v_n(x,\xi) \right) d\xi,$$

Now we can start with initial approximation $u_0(x,0) = \sinh x$, $v_0(x,0) = \cosh x$ and using the iteration formula ( ), we obtained following successive approximation

$u_0 = u(x,0) = \sinh x, \qquad v_0 = v(x,0) = \cosh x,$

$u_1 = \sinh x + t \cosh x,$

$v_1 = \cosh x + t \sinh x,$

$u_2 = \sinh x + t \cosh x + \frac{t^2}{2} \sinh x,$

$v_2 = \cosh x + t \sinh x + \frac{t^2}{2} \cosh x,$

$\vdots$

$u_n = \sinh x \left(1 + \frac{t^2}{2!} + \cdots \right) + \cosh x \left(t + \frac{t^3}{3!} + \cdots \right),$

$v_n = \cosh x \left(1 + \frac{t^2}{2!} + \cdots \right) + \sinh x \left(t + \frac{t^3}{3!} + \cdots \right),$

When $n \to \infty$ then

$$(u,v) = \lim_{n\to\infty} (u_n, v_n)$$

$(u,v) = (\sinh x \cosh t + \cosh x \sinh t \, , \cosh x \cosh t + \sinh x \sinh t)$

$$(u,v) = (\sinh(x+t) \, , \cosh(x+t))$$

Example (2). Consider the following linear system:

$u_t + u_x - 2v = 0$

$v_t + v_x + 2u = 0$

$u(x,0) = \sin x \, , \qquad v(x,0) = \cos x$

Solution : Let’s consider the linear equations :

$u_t + u_x - 2v = 0, \;\; u(x,0) = \sin x$

$v_t + v_x + 2u = 0, \;\; v(x,0) = \cos x$

Then the correction functional is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - 2v_n(x,\xi)\right) d\xi,$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} + 2u_n(x,\xi)\right) d\xi,$$

We can select $u_0(x,0) = \sin x$, $v_0(x,0) = \cos x$ by using the given initial values. Accordingly, we obtain the following successive approximations

$u_0 = u(x,0) = \sin x,\qquad v_0 = v(x,0) = \cos x,$

$u_1 = \sin x + t\cos x,$

$v_1 = \cos x - t\sin x,$

$u_2 = \sin x + t\cos x - \frac{t^2}{2}\sin x,$

$v_2 = \cos x - t\sin x - \frac{t^2}{2}\cos x,$

⋮

$u_n = \sin x + t\cos x - \frac{t^2}{2!}\sin x + \cdots$

$v_n = \cos x - t\sin x - \frac{t^2}{2}\cos x + \cdots$

When $n \to \infty$ then

$$(u,v) = (\sin x\cos t + \cos x\sin t\, , \cos x\cos t - \sin x\sin t)$$

$$(u,v) = (\sin(x+t)\, , \cos(x+t))$$

Example (3). Consider the linear system:

$u_t + u_x - 2v_x = 0$

$v_t + v_x - 2u_x = 0$

$u(x,0) = \cos x\, ,\qquad v(x,0) = \cos x$

Solution : the correction functional is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - 2\frac{\partial v_n(x,\xi)}{\partial x}\right) d\xi,$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} - 2\frac{\partial u_n(x,\xi)}{\partial x}\right) d\xi,$$

We can select $u_0(x,0) = \cos x$, $v_0(x,0) = \cos x$ by using the given initial values. Accordingly, we obtain the following successive approximations

$u_0 = u(x,0) = \cos x,\qquad v_0 = v(x,0) = \cos x,$

$u_1 = \cos x - t\sin x,$

$v_1 = \cos x - t\sin x,$

$u_2 = \cos x - t\sin x - \frac{t^2}{2}\cos x,$

$v_2 = u_2,$

$u_3 = \cos x - t\sin x - \frac{t^2}{2}\cos x + \frac{t^3}{6}\sin x,$

$v_3 = u_3,$

$\vdots$

$u_n = v_n = \cos x\left(1 - \frac{t^2}{2!} + \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \cdots\right),$

When $n \to \infty$ then

$(u,v) = (\cos x\cos t - \sin x\sin t\,, \cos x\cos t - \sin x\sin t)$

$$(u,v) = (\cos(x+t)\,, \cos(x+t))$$

Example (4). Consider the following system

$$u_t - v_x + (u+v) = 0$$

$$v_t - u_x + (u+v) = 0$$

$u(x,0) = \sinh x\,,\ \ v(x,0) = \cosh x,$

Solution :

the correction functional is given by

$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial v_n(x,\xi)}{\partial x} + u_n(x,\xi) + v_n(x,\xi)\right) d\xi,$

$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - \frac{\partial u_n(x,\xi)}{\partial x} + u_n(x,\xi) + v_n(x,\xi)\right) d\xi,$

We can select $u_0(x,0) = \sinh x$, $v_0(x,0) = \cosh x$, by using the given initial values. Accordingly, we obtain the following successive approximations

$u_0 = u(x,0) = \sinh x\,,\ \ v_0 = v(x,0) = \cosh x,$

$u_1 = \sinh x - t\cosh x,$

$v_1 = \cosh x - t\sinh x,$

$u_2 = \sinh x - t\cosh x + \frac{t^2}{2}\sinh x,$

$v_2 = \cosh x - t\sinh x + \frac{t^2}{2}\cosh x,$

$\vdots$

$u_n = \sinh x\left(1 + \frac{t^2}{2!} + \cdots\right) - \cosh x\left(t + \frac{t^3}{3!} + \cdots\right),$

$v_n = \cosh x\left(1 + \frac{t^2}{2!} + \cdots\right) - \sinh x\left(t + \frac{t^3}{3!} + \cdots\right),$

When $n \to \infty$ then

$$(u, v) = (\sinh x\cosh t - \cosh x\sinh t\,, \cosh x\cosh t - \sinh x\sinh t)$$

$$(u, v) = (\sinh(x - t)\,, \cosh(x - t))$$

Example (5). Consider the Nonlinear system

$u_t + u_x v_x = 2$

$v_t + u_x v_x = 0$

$u(x, 0) = x, \quad v(x, 0) = x$

Solution :

the correction functional is given by

$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x}.\frac{\partial v_n(x,\xi)}{\partial x} - 2\right) d\xi,$

$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x}.\frac{\partial v_n(x,\xi)}{\partial x}\right) d\xi,$

We can select $u_0(x, 0) = x,\ v_0(x, 0) = x$, by using the given initial values. Accordingly, we obtain the following successive approximations

$u_0 = u(x, 0) = x,\ v_0 = v(x, 0) = x,$

$u_1 = x + t,$

$v_1 = x - t,$

$u_2 = x + t,$

$v_2 = x - t,$

$\vdots$

$u_n = x + t,$

$v_n = x - t,$

When $n \to \infty$ then

$$(u, v) = (x + t, x - t)$$

Example (6): Consider the following Nonlinear system

$u_t + 2vu_x - u = 2$

$v_t - 3uv_x + v = 3$

$u(x, 0) = e^x, \ \ v(x, 0) = e^{-x}$

Solution : the correction functional is given by

$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + 2v_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - u_n(x,\xi) - 2\right) d\xi,$

$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - 3u_n(x,\xi)\frac{\partial v_n(x,\xi)}{\partial x} - 3\right) d\xi,$

We can select $u_0(x, 0) = e^x, \ v_0(x, 0) = e^{-x}$, by using the given initial values. Accordingly, we obtain the following successive approximations

$u_0 = u(x, 0) = e^x, \quad v_0 = v(x, 0) = e^{-x},$

$u_1 = e^x + te^x,$

$v_1 = e^{-x} - te^{-x},$

$u_2 = e^x + te^x + \frac{t^2}{2}e^x + \frac{2}{3}t^3,$

$v_2 = e^{-x} - te^{-x} + \frac{t^2}{2}e^{-x} + t^3,$

$\vdots$

$u_n = e^x\left(`1 + t + \frac{t^2}{2} + \cdots\right),$

$v_n = e^{-x}\left(1 - t + \frac{t^2}{2} + \cdots\right),$

When $n \to \infty$ then

$$(u, v) = (e^{x+t}, e^{x-t})$$

V. CONCLUSION

The linear partial differential equations and nonlinear partial differential equations have been solved by variational iteration method , this method is very effective and accelerate the convergent of solution ,the study showed that this method is easy to apply and it is more accurate and effective.

REFERENCES:

[1]. A .A. Hemeda, Variational iteration method for solving wave equation, Computers and Mathematics with Applications,2008, 1948–1953.

[2]. Abdul-Majid Wazwaz , A comparison between the variational iteration method and Adomian decomposition Method, Journal of Computational and Applied Mathematics, 2007.

[3]. B. A. Finlayson , The Method of Weighted Residuals and Variational Principles, Academic Press, New York, 1972.

[4]. Chen W, Zhang XD, Koroša D, Investigation on fractional relaxation-oscillation models, , International Journal of Nonlinear Sciences and Numerical Simulation.

[5]. Chen W. Chaos,Time-space fabric underlying anomalous diffusion , Solitons and Fractals , 2006.

[6]. Draganescu GE , Application of a variational iteration method to linear and nonlinear viscoelastic models with fractional derivatives, Journal of Mathematical Physics , 2006.

[7]. J.H. He, Generalized Variational Principles in Fluids, , Science & Culture Publishing House of China, 2003 .

[8]. J.H. He, Generalized Variational Principles in Fluids, , Science & Culture Publishing House of China, 2003.

[9]. Ji-Huan He, Guo-Cheng Wu, F. Austin, The Variational Iteration Method Which Should Be Followed, Nonl. Sci. Lett. A, 2010.