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Solutions of systems of linear partial differential equations and nonlinear partial differential equations by using Variational Iteration Method

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Abstract
In this paper, the possibility of application of the variational iteration method for solving the systems of linear PDEs and nonlinear PDEs subject to the general initial conditions by using THE variational iteration method. four illustrated examples has been introduced. The steps of the method are easy implemented and high accuracy.
Keywords: *System of linear partial differential equations, system of nonlinear partial differential equations, variational iteration method.*

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I. INTRODUCTION

The method VIM is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations, which converge rapidly to the accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

The variational iteration method (VIM) is relatively new approaches to provide approximate solutions to linear and nonlinear problems. The variational iteration method, (VIM) was successfully applied to find the solutions of several classes of variational problems.

Here we used VIM for solving systems of linear or nonlinear PDEs with initial conditions. This paper is arranged as follows. In section 2 General Lagrange multiplier and VIM. . In section 3, Systems of Linear PDEs by Variational Iteration Method. In section 4, Systems of Nonlinear PDEs by Variational Iteration Method In section 5 numerical examples. The conclusions in section 6.

II. GENERAL LAGRANGE MULTIPLIER and VIM

Lagrange multiplier is well known in optimization and calculus of variations. In order to understand the concept of the general Lagrange multiplier, we consider the differential equation

$$Lu + Nu = g(t), \qquad (1)$$

where L and N are linear and nonlinear operators respectively, and g(t) is the source inhomogeneous term.

The variational iteration method presents a correction functional for Eq.(1) in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)\big(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\big)d\xi, \qquad (2)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of the He 's variational iteration method require first the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally. Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use

$$\int \lambda(\xi)u_n'(\xi)d\xi = \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi,$$

$$\int \lambda(\xi)u_n''(\xi)d\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi,$$

and so on. The last two identities can be obtained by integrating by parts.

III. SYSTEM of LINEAR PDEs by VARIATIONAL ITERATION METHOD

In this section we will apply the variational iteration method for solving systems of linear partial differential equations. We write a system in an operator form by

$$\begin{aligned} L_t u + R_1(u,v) &= g_1, \\ L_t v + R_2(u,v) &= g_2, \end{aligned} \qquad (3)$$

Where $u = u(x,t)$, with initial data

$$u(x,0) = f_1(x),$$

$$v(x,0) = f_2(x),$$

where L_t is considered a first order partial differential operator, and $R_j, 1 \leq j \leq 3$ are linear operators, and g_1, and g_2 are source terms. Following the discussion presented above for variational iteration method, the following correction functionals for the system (3) can be set in the form

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1\big(Lu_n(\xi) + R_1(\tilde{u}_n, \tilde{v}_n) - g_1(\xi)\big)d\xi \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2\big(Lv_n(\xi) + R_2(\tilde{u}_n, \tilde{v}_n) - g_2(\xi)\big)d\xi \end{aligned} \qquad (4)$$

where $\lambda_j\ , j = 1{,}2$ are general Lagrange multipliers, which can be identified optimally via the variational theory, and $\tilde{u}_n$, and $\tilde{v}_n$ as restricted variations which means $\delta\tilde{u}_n = 0$, and $\delta\tilde{v}_n = 0$. The Lagrange multipliers $\lambda_j\ , j = 1{,}2$ will be identified optimally via integration by parts as introduced before. The successive approximations$u_{n+1}(x,t)$ and $v_{n+1}(x,t), n \geq 0$, of the solutions $u(x,t)$ and $v(x,t)$ will follow immediately upon using the obtained Lagrange multipliers and by using selective functions u_0 and v_0. The initial values may be used for the selective zeroth approximations. With the Lagrange multipliers λ_j determined, several approximations $u_j(x,t), v_j(x,t), j \geq 0$ can be computed. Consequently, the solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t) \end{aligned} \qquad (5)$$

To give a clear overview of the analysis introduced above, the two examples that were studied before will be used to explain the technique that we summarized before, therefore we will keep the same numbers.

IV. SYSTEMS of NONLINEAR PDEs by VARIATIONAL ITERATION METHOD

Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of chemical reaction-diffusion model. To use the VIM, we write a system in an operator form by

$$\begin{aligned} L_t u + R_1(u,v,w) + N_1(u,v,w) &= g_1, \\ L_t v + R_2(u,v,w) + N_2(u,v,w) &= g_2, \\ L_t w + R_3(u,v,w) + N_3(u,v,w) &= g_3, \end{aligned} \qquad (6)$$

with initial data

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x)\,, \\ w(x,0) &= f_3(x), \end{aligned} \qquad (7)$$

where L_t is considered a first order partial differential operator, and $R_j, 1 \leq j \leq 3$ and $N_j, 1 \leq j \leq 3$ are linear and nonlinear operators respectively, and g_1, g_2 and g_3 are source terms. The correction functionals for equations of the system (7) can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1\big(Lu_n(x,\xi) + R_1(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_1(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_1(\xi)\big)d\xi,$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2\big(Lv_n(x,\xi) + R_2(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_2(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_2(\xi)\big)d\xi$$

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3\big(Lw_n(x,\xi) + R_3(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_3(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_3(\xi)\big)d\xi \qquad (8)$$

where λ_j ,$1 \leq j \leq 3$ are general Lagrange 's multipliers, which can be identified optimally via the variational theory, and $\tilde{u}_n, \tilde{v}_n$ and $\tilde{w}_n$ as restricted variations which means $\delta\tilde{u}_n = 0, \delta\tilde{v}_n = 0$ and $\delta\tilde{w}_n = 0$. It is required first to determine the Lagrange multipliers λ_j that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$,$v_{n+1}(x,t)$ and $w_{n+1}(x,t), n \geq 0$, of the solutions $u(x,t), v(x,t)$ and $w(x,t)$ will follow immediately upon using the obtained Lagrange multipliers and by using selective functions u_0, v_0 and w_0. The initial values are usually used for the selective zeroth approximations. With the Lagrange multipliers λ_j determined, then several approximations $u_j(x,t), v_j(x,t), w_j(x,t), j \geq 0$ can be computed. Consequently, the solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t) \\ w(x,t) &= \lim_{n\to\infty} w_n(x,t) \end{aligned} \qquad (9)$$

To give a clear overview of the analysis introduced above, we will apply the VIM to the same two illustrative systems of partial differential equations that were studied in the previous section..

V. NUMERICAL EXAMPLES

Example(1) We first consider the linear system:

$$u_x + v_y - w_t = 1$$

$$v_x + w_y + u_t = 1$$

$$w_x + u_y + v_t = 1$$

with the initial data

$$u(x,y,0) = x + y, \qquad v(x,y,0) = x + y, \quad w(x,y,0) = x - y$$

Where $u = u(x,y,t), v = v(x,y,t)$ and $w = w(x,y,t)$

Solution :

$w_t - u_x - v_y + 1 = 0,$

$u_t + v_x + w_y - 1 = 0,$

$v_t + u_y + w_x - 1 = 0,$

The correction functionals

$$u_{n+1}(x,y,t) = u_n + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n}{\partial \xi} + \frac{\partial v_n}{\partial x} + \frac{\partial w_n}{\partial y} - 1\right) d\xi,$$

$$v_{n+1}(x,y,t) = v_n + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n}{\partial \xi} + \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial x} - 1\right) d\xi,$$

$$w_{n+1}(x,y,t) = w_n + \int_0^t \lambda_3(\xi)\left(\frac{\partial w_n}{\partial \xi} - \frac{\partial u_n}{\partial x} - \frac{\partial v_n}{\partial y} + 1\right) d\xi,$$

This gives the stationary conditions

$$1 + \lambda_1|_{\xi=t} = 0,$$

$$\lambda_1'(\xi = t) = 0,$$

and

$$1 + \lambda_2|_{\xi=t} = 0,$$

$$\lambda_2'(\xi = t) = 0,$$

and

$$1 + \lambda_3|_{\xi=t} = 0,$$

$$\lambda_3'(\xi = t) = 0,$$

As a result we find

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

Substituting these values of the Lagrange multipliers into the above functionals gives the iteration formulas :

$$u_{n+1}(x,y,t) = u_n - \int_0^t \left(\frac{\partial u_n}{\partial \xi} + \frac{\partial v_n}{\partial x} + \frac{\partial w_n}{\partial y} - 1\right) d\xi,$$

$$v_{n+1}(x,y,t) = v_n - \int_0^t \left(\frac{\partial v_n}{\partial \xi} + \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial x} - 1\right) d\xi,$$

$$w_{n+1}(x,y,t) = w_n - \int_0^t \left(\frac{\partial w_n}{\partial \xi} - \frac{\partial u_n}{\partial x} - \frac{\partial v_n}{\partial y} + 1\right) d\xi,$$

We can select $u_0, v_0 and\ w_0$ by using the given initial values:

$u_0 = u(x,y,0) = x + y,$

$v_0 = v(x,y,0) = x + y,$

$w_0 = w(x,y,0) = x - y,$

Accordingly, we obtain the following successive approximations
$u_0 = x + y,$

$v_0 = x + y,$

$w_0 = x - y,$

$u_1 = x + y + t,$

$v_1 = x + y - t,$

$w_1 = x - y + t,$

$u_2 = x + y + t,$

$v_2 = x + y - t,$

$w_2 = x - y + t,$

$\vdots$

$u_n = x + y + t,$

$v_n = x + y - t,$

$w_n = x - y + t,$

When $n \to \infty$ then

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (x + y + t, x + y - t, x - y + t)$$

Example (2) We first consider the linear system:

$$\begin{aligned} u_x + v_t + w_y &= e^x \\ v_y + w_x + u_t &= e^y \\ w_t + u_y + v_x &= e^t \end{aligned}$$

with the initial data :

$u(x,y,0) = e^x, v(x,y,0) = e^y, w(x,y,0) = 1$

Solution :

$v_t + u_x + w_y - e^x = 0,$
$u_t + v_y + w_x - e^y = 0,$
$w_t + u_y + v_x - e^t = 0,$

The correction functionals

$$u_{n+1}(x,y,t) = u_n + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n}{\partial \xi} + \frac{\partial v_n}{\partial y} + \frac{\partial w_n}{\partial x} - e^y\right) d\xi,$$

$$v_{n+1}(x,y,t) = v_n + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n}{\partial \xi} + \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} - e^x\right) d\xi,$$

$$w_{n+1}(x,y,t) = w_n + \int_0^t \lambda_3(\xi)\left(\frac{\partial w_n}{\partial \xi} + \frac{\partial u_n}{\partial y} + \frac{\partial v_n}{\partial x} - e^\xi\right) d\xi,$$

This gives the stationary conditions

$$1 + \lambda_1|_{\xi=t} = 0,$$

$$\lambda_1'(\xi = t) = 0,$$

and

$$1 + \lambda_2|_{\xi=t} = 0,$$

$$\lambda_2'(\xi = t) = 0,$$

and

$$1 + \lambda_3|_{\xi=t} = 0,$$

$$\lambda_3'(\xi = t) = 0,$$

As a result we find

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

Substituting these values of the Lagrange multipliers into the above functionals gives the iteration formulas :

$$u_{n+1}(x,y,t) = u_n - \int_0^t \left(\frac{\partial u_n}{\partial \xi} + \frac{\partial v_n}{\partial y} + \frac{\partial w_n}{\partial x} - e^y\right) d\xi,$$

$$v_{n+1}(x,y,t) = v_n - \int_0^t \left(\frac{\partial v_n}{\partial \xi} + \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} - e^x\right) d\xi,$$

$$w_{n+1}(x,y,t) = w_n - \int_0^t \left(\frac{\partial w_n}{\partial \xi} + \frac{\partial u_n}{\partial y} + \frac{\partial v_n}{\partial x} - e^\xi\right) d\xi,$$

We can select $u_0, v_0 and\ w_0$ by using the given initial values:

$u_0 = u(x,y,0) = e^x,$

$v_0 = v(x,y,0) = e^y,$

$w_0 = w(x,y,0) = 1,$

Accordingly, we obtain the following successive approximations

$u_0 = e^x$

$v_0 = e^y$

$w_0 = 1$

$u_1 = e^x$

$v_1 = e^y$

$w_1 = 1 + e^t$

$u_2 = e^x$

$v_2 = e^y$

$w_2 = 1 + e^t$

$\vdots$

$u_n = e^x$

$v_n = e^y$

$w_n = 1 + e^t$

When $n \to \infty$ then

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (e^x, e^y, 1 + e^t)$$

Example (3) Consider the nonlinear system of partial differential equation

$$u_t + v_x w_y - v_y w_x = -u$$

$$v_t + w_x u_y + w_y u_x = v$$

$$w_t + u_x v_y + u_y v_x = w$$

with the initial data

$$u(x, y, 0) = e^{x+y}, \;\; v(x, y, 0) = e^{x-y}, \qquad w(x, y, 0) = e^{-x+y}$$

Solution :

Firstly we write the correction functionals

$$u_{n+1} = u_n + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n}{\partial \xi} + \frac{\partial v_n}{\partial x}.\frac{\partial w_n}{\partial y} - \frac{\partial v_n}{\partial y}.\frac{\partial w_n}{\partial x} + u_n\right) d\xi$$

$$v_{n+1} = v_n + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n}{\partial \xi} + \frac{\partial w_n}{\partial x}.\frac{\partial u_n}{\partial y} + \frac{\partial w_n}{\partial y}.\frac{\partial u_n}{\partial x} - v_n\right) d\xi$$

$$w_{n+1} = w_n + \int_0^t \lambda_3(\xi)\left(\frac{\partial w_n}{\partial \xi} + \frac{\partial u_n}{\partial x}.\frac{\partial v_n}{\partial y} + \frac{\partial u_n}{\partial y}.\frac{\partial v_n}{\partial x} - w_n\right) d\xi$$

This gives the stationary conditions

$$1 + \lambda_1|_{\xi=t} = 0,$$

$$\lambda_1'(\xi = t) = 0,$$

and

$$1 + \lambda_2|_{\xi=t} = 0,$$

$$\lambda_2'(\xi = t) = 0,$$

and

$$1 + \lambda_3|_{\xi=t} = 0,$$

$$\lambda_3'(\xi = t) = 0,$$

As a result we find

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

Substituting these values of the Lagrange multipliers into the above functionals gives the iteration formulas :

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n}{\partial \xi} + \frac{\partial v_n}{\partial x}.\frac{\partial w_n}{\partial y} - \frac{\partial v_n}{\partial y}.\frac{\partial w_n}{\partial x} + u_n\right) d\xi$$

$$v_{n+1} = v_n - \int_0^t \left(\frac{\partial v_n}{\partial \xi} + \frac{\partial w_n}{\partial x}.\frac{\partial u_n}{\partial y} + \frac{\partial w_n}{\partial y}.\frac{\partial u_n}{\partial x} - v_n\right) d\xi$$

$$w_{n+1} = w_n - \int_0^t \left(\frac{\partial w_n}{\partial \xi} + \frac{\partial u_n}{\partial x}.\frac{\partial v_n}{\partial y} + \frac{\partial u_n}{\partial y}.\frac{\partial v_n}{\partial x} - w_n\right) d\xi$$

The zeroth approximations $u_0, v_0 and\ w_0$ are selected by using the given initial conditions. Therefore, we obtain the following successive approximations

$u_0 = u(x, y, 0) = e^{x+y}$,

$v_0 = v(x, y, 0) = e^{x-y}$,

$w_0 = w(x, y, 0) = e^{-x+y}$,

$u_1 = e^{x+y} - te^{x+y}$,

$v_1 = e^{x-y} + te^{x-y}$,

$w_1 = e^{-x+y} + te^{-x+y}$,

$u_2 = e^{x+y} - te^{x+y} + \frac{t^2}{2} e^{x+y}$,

$v_2 = e^{x-y} + te^{x-y} + \frac{t^2}{2} e^{x-y}$,

$w_2 = e^{-x+y} + te^{-x+y} + \frac{t^2}{2} e^{-x+y}$,

$\vdots$

When $n \to \infty$ then

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (e^{x+y-t}, e^{x-y+t}, e^{-x+y+t})$$

Example (4) Consider the nonlinear system of partial differential equation

$$\begin{aligned} u_t + u_x v_x - w_y &= 1 \\ v_t + v_x w_x + u_y &= 1 \\ w_t + w_x u_x - v_y &= 1 \end{aligned}$$

with initial data

$u(x, y, 0) = x + y, v(x, y, 0) = x - y, w(x, y, 0) = -x + y$

Solution :

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n}{\partial \xi} + \frac{\partial u_n}{\partial x} . \frac{\partial v_n}{\partial x} - \frac{\partial w_n}{\partial y} - 1 \right) d\xi$$

$$v_{n+1} = v_n - \int_0^t \left(\frac{\partial v_n}{\partial \xi} + \frac{\partial v_n}{\partial x} . \frac{\partial w_n}{\partial x} + \frac{\partial u_n}{\partial y} - 1 \right) d\xi$$

$$w_{n+1} = w_n - \int_0^t \left(\frac{\partial w_n}{\partial \xi} + \frac{\partial w_n}{\partial x} . \frac{\partial u_n}{\partial x} - \frac{\partial v_n}{\partial y} - 1 \right) d\xi$$

The zeroth approximations $u_0, v_0 and\ w_0$ are selected by using the given initial conditions. Therefore, we obtain the following successive approximations

$u_0 = u(x, y, 0) = x + y,$

$v_0 = v(x, y, 0) = x - y,$

$w_0 = w(x, y, 0) = -x + y,$

$u_1 = x + y + t,$

$v_1 = x - y + t,$

$w_1 = -x + y + t,$

$u_2 = x + y + t,$

$v_2 = x - y + t,$

$w_2 = -x + y + t,$

$\vdots$

$u_n = x + y + t,$

$v_n = x - y + t,$

$w_n = -x + y + t,$

when $n \to \infty$ then

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (x + y + t, x - y + t, -x + y + t)$$

VI. CONCLUSION

The linear partial differential equations and nonlinear partial differential equations have been solved by variational iteration method , this method is very effective and accelerate the convergent of solution ,the study showed that this method is easy to apply and it is more accurate and effective.

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