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# On Affine Spaces

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***Abstract***
*The aim of this paper is to review the area of affine spaces. Starting from the notation of vector space, we define an affine space by a set of axioms given by H. Weyl and study basic concepts such as affine coordinate systems, affine subspaces and affine transformations. After treating various ways of representing affine subspaces (including Barry centric coordinates), we discuss convex sets in a real affine space.*

***Keywords:*** *Affine space, Convex set.*

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## I. INTRODUCTION

Affine space [1] deals with linear algebra. The concept of affine spaces is more similar to vector space. In vector space the elements were called vectors while in an affine space, the elements are called points. Many problems related to linear algebra require geometric facts associated with mutual positions of figures. This can be met by using affine spaces.

In 1748 Euler introduced the term Affine in his book referred to affine geometry for his text space, time and matter. He uses affine geometry to introduce vector addition and subtraction at the earliest stages of his development of mathematical physics. Weyl's geometry [2] is interesting historically as having been the first of affine geometry to be worked out in detail.

Geometrically, curves and surfaces are usually considered to sets of points with some special properties, living in a space consisting of points. Typically, one is also interested in geometric properties invariant under certain transformation, for e.g. translations, notations, projections, etc. One could model the space of points as a vector space but this is not very satisfactory for a number of reasons. One reason is that the point corresponding to the zero vector (0) called the origin, plays a special role, when there is really no reason to have a privileged origin. Another reason is that certain notations, such as parallelism, are handled in an awkward manner. But the deeper reason is that vector space and affine space really have different geometries. The geometric properties of a vector space are invariant under group of bijective affine maps, and these two groups are not isomorphic. Roughly speaking, there are more affine maps than linear maps. Affine spaces are the right frame work for dealing with motions, trajectories, and physical forces, among other things. Thus, affine geometry [3] is crucial to a clean presentation of kinematics, dynamics and other parts of physics.

The concept of affine spaces has its roots in the evolution of geometry, beginning with classical Euclidean geometry and evolving into a more abstract framework in the 19th and 20th centuries. The foundation of affine spaces can be traced back to Euclidean geometry (c. 300 BCE), as described in Elements by Euclid. While Euclid dealt primarily with metric geometry (distances and angles), some ideas, such as parallelism and proportionality, hinted at concepts central to affine geometry. The introduction of coordinate geometry by René Descartes and Pierre de Fermat in the 17th century laid the groundwork for connecting algebra and geometry. This connection was crucial for later developments of affine spaces, as it allowed geometric objects to be represented using equations. Mathematicians like Leonhard Euler and Joseph-Louis Lagrange [4] studied transformations that preserve certain geometric properties, such as parallelism. Jean-Victor Poncelet (early 19th century) formalized ideas in projective geometry, closely related to affine geometry, as a framework for studying geometric properties invariant under projection. Möbius [5] introduced barycentric coordinates in 1827, which became a key tool for working with points in affine spaces. Von Staudt [6] distinguished affine geometry from projective geometry and emphasized its independence from metric concepts like distance and angle. Klein's Erlangen Program categorized geometries based on the invariants of transformation groups [7]. Affine geometry was defined as the study of properties invariant under affine transformations, such as parallelism. The formalization of vector spaces and linear algebra provided a solid foundation for affine spaces. Élie Cartan [8,9] and others expanded the study of affine geometry to include connections and curvature in differential geometry.

## II. DEFINITION AND PROPERTIES

Let X be a collection whose elements are called points, denoted by capital letters A, B, ……….., M. Also let V be a vector space of finite dimension over an arbitrary field F.

**Definition 1**

A non-empty set X is called an affine space associated to V if there is a mappingof X x X into V, denoted by $(P,Q) \to \overrightarrow{(PQ)}$

$$X \times X \to V.$$

Which has the following properties;

a) for any three points P, Q, R in X ,we have $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

b) for any P ϵ X and for any α ϵ V there is one and only one point Q ϵ X such that $\overrightarrow{PQ} = \alpha$

**Remark 1**

$\overrightarrow{PQ}$ is the vector determined by the initial point P and the end point Q. Property a) is known as Chasles's identity. An affine space is real or complex, if the corresponding vector space V is real or complex.

**EXAMPLES**

- Every vector space V defines an affine space (V,V)

Let a, b ϵ V considered as points of the set X = V. If we set a b = b-a, We can easily see that it satisfies all the axioms of affine space. Thus (V,V) is an affine space.

- $R^2$ ,$R^3$ are affine spaces.

In general $R^n$ is an affine space for all n.

- If (X, U) and (Y, V) are affine spaces, then (X x Y, U x V) is again an affine space

**Definition 2**

An ordered (n+1) – tuple of points {$P_0$ , $P_1$ ………..$Pn$ } in an affine space X is called affine frame if the vectors $P_0$ $P_1$ , 1≤i≤n, form a basis of V. The point $P_0$ is called the origin, and $P_i$ the ith unit point of the affine frame

**Definition 3**

The dimension of the affine space (X, V) is the dimension of the associated vector space V.

**Remark 2**

1. The affine space is finite or infinite dimensional as V is finite or infinite dimensional.
2. An affine space of dimension 1 is called an affine line.
3. An affine space of dimension 2 is called an affine frame.
4. $R^2$ is an affine space of dimension 2.
5. $R^3$ is an affine space of dimension 3.

In general $R^n$ is an affine space of dimension n.

**Definition 4**

Let (X, V) be an n dimensional affine space, here we introduce the affine system of coordinates. For that, let O ϵ X be an arbitrary point, called origin. Let {$e_1$ $e_2$ …….en} be the basis of V. Let M be an arbitrary point in X. Define vector $OM$ ϵ V called radius vector of M by $OM = \alpha_1 e_1 + \alpha_2 e_2 + \ldots \ldots \alpha n\, en$. The ordered collection of coefficients ($\alpha_1$ , $\alpha_2$ ,…… αn) is called the affine coordinates or Bary centric coordinates of M. Moreover, the point O and basis {$e_1$ $e_2$ …….en} together are called a frame of reference in space V usually denoted by (O: $e_1$ $e_2$ …….en).

**Remark 3:**

Let N be another point of the same affine space with coordinates {$\beta_1$ ,$\beta_2$ ……βn} corresponding to the same frame of reference. ie; [$ON = \beta_1 e_1 + \beta_2 e_2 + \ldots\ldots + \beta n\, en$].Then the vector $\overrightarrow{MN}$ can be expressed in terms of affine coordinates of M and N by $MN = MO + ON$

$$= -OM + ON$$

$$= ON - OM$$

$= [\beta_1 e_1 + \beta_2 e_2 + \cdots \ldots \ldots . . +\beta nen] - [\alpha_1 e_1 + \alpha_2 e_2 + \cdots \ldots \ldots \alpha nen]$
$= \beta_1 e_1 + \beta_2 e_2 + \ldots\ldots\ldots + \beta nen - \alpha_1 e_1 - \alpha_2 e_2 - \ldots\ldots\ldots - \alpha nen$
$= (\beta_1 - \alpha_1)e_1 + (\beta_2 - \alpha_2)e_2 + \ldots\ldots\ldots + (\beta n - \alpha n)en$

Thus $MN$ has co-ordinates ($\beta_1 - \alpha_1$ , $\beta_2 - \alpha_2$ ,…….βnαn) ,i.e., one can obtain the coordinates of $MN$ by subtracting the co-ordinates of respective points with respect to origin.

The affine co-ordinates of a point depends on the fixed point 0 as well as the considered basis of V. Also, if we fix a point 0 and {$e_1$ , $e_2$ , ……en}. Then the affine co-ordinate of a point are unique. As every vector of a space is determined by it's co-ordinates . Similarly every point in an affine space is determined by it's co-ordinates with respect the given frame of reference. Thus the role of frame of reference in an affine space is equivalent to the role of a basis in vector space. The vectors {$e_1$ , $e_2$ , ……en} can be written in the form ei = $OAi$ and thus we can write the frame of reference as {O,$A_1$ ,$A_2$ ………,An} under the condition $\{OA1, OA2 , \ldots \ldots . . OAn\}$ form a basis of V .[i.e. they must be linearly independent]. In Euclidian geometry, Cartesian co-ordinates are affine co-ordinates relative to an orthonormal frame, that is an affine frame (o, $v_1$ , $v_2$ , ….vn) such that ($v_1$ , $v_2$ ,……vn) is an orthonormal basis.

**Definition 5**
A non-empty subset Y of X is called an affine subspace of X, if for some P ϵ Y, the set of vectors
WP(Y) = {$PQ$ : Q ϵ Y} is a subspace of V.
This definition is actually independent of the choice of P.

**Proposition 1**

If for some P ϵ Y ,the set Wp(Y) is a subspace of V, then for any P'ϵ Y, the set Wp'(Y) = {$\vec{P'}\vec{Q}$ ;Q ϵ Y} is a subspace of V and in fact Wp'(Y) = WP(Y).

**Proof**
We shall show that Wp'(Y) = WP(Y). For any Q ϵ Y
.We have $\vec{P'}\vec{Q} = \vec{P'}\vec{P} + PQ$
$= -\vec{P}\vec{P'} + PQ$ [ by prop.2.1], where $\vec{P'}\vec{P}$ , $\vec{PQ}$ ϵ WP(Y).
Since WP(Y) is a subspace of V, we see that $\vec{P'}\vec{Q}$ ϵ WP(Y).
This proves $Wp$,(Y) ⊆ $Wp$(Y).
In order to prove, $Wp$(Y) ⊆ $Wp'$ (Y). let α ϵ WP(Y).
If we denote $pp$' ϵ WP(Y) by β, then α+β is in the subspace of WP(Y).
Let Q' be a point of Y such that $PQ$' = α+β.
Then, $PQ$'= $PP$' + $P$'$Q$'
this implies α+β = β + $P$'$Q$' ,
i.e. $P$'$Q$'= α, showing that α is in Wp'(Y). Since α is an arbitrary element of Wp(Y),
we have Wp(Y) ⊂ WP'(Y). Hence Wp'(Y) = WP(Y)

**Definition 6**
For an affine subspace Y of X, the vector space Wp(Y) which is independent of P ϵ Y, is called the vector space associated with Y. we shall denote it simply by W(Y). We define the dimension of Y to be dim W(Y).

**Remark 4**
(Y, Wp(Y)) is itself an affine space.Affine subspace is sometimes called an flat.

**Definition 7**
An affine subspace Y dimension 0 is a subset consisting of one single point. An affine subspace Y is called a line, plane , hyper plane according as dim Y = 1, 2, n-1, where n = dim X. (For n = 2 ,a line and a hyper plane mean the same thing, and for n=3,a plane and a hyper plane mean the same thing).

**Proposition 2**
Let $Y_\lambda$ be a family of affine subspaces of an affine space X. If $\cap Y_\lambda$ is not empty, it is an affine subspace associated with $\cap W_\lambda$, where $W_\lambda$ is the vector space associated with $Y_\lambda$.
**Proof:**
Let pϵ $\cap Y_\lambda$. For any Q ϵ $\cap Y_\lambda$.
We have $PQ$ϵ $W_\lambda$ ; □ λ.

And hence $PQ \epsilon \cap W_\lambda$.
Conversely,
given any α ϵ $\cap W_\lambda$, there exists for each λ, a point $Q_\lambda \in \cap W_\lambda$
such that $PQ_\lambda$=α. Since there is one and only one point Q ϵ X such that $PQ$= α ,
we must have $Q_\lambda$ = Q □ λ.
Thus ,
Q ϵ $\cap Y_\lambda$ and $PQ$= α , showing that
$\cap Y_\lambda$ = {Q ϵ X :$PQ \epsilon \cap W_\lambda$.}
That is, ϵ $\cap Y_\lambda$ is an affine subspace associated with $\cap Y_\lambda$.

**Definition 8**
Let S be a subspace of a vector space V . The coset v + S = {v + s/ s ϵ S}is called a flat in V with base S and flat representative v . We also refer to v + S a translate of S. The set A(V) of all flats in V is called the affine geometry of V. The dimension dim (A (V)) of A(V) is defined to be dim(V).While a flat may have many flat representatives, it only has one base since x + S = y + T implies that x ϵ y + T and so x + S = y + T = x + T .whence S = T.

**Definition 9**
Two flats X = x + S and Y = y + T are said to be parallel if $S \subseteq T$ or $T \subseteq S$. This denoted by $X \| Y$.

We will denote subspaces of by the letters S, T…… and flats in V by X, Y……Here are some of the basic intersection properties of flats.

**Theorem 1:**

Let S and T be subspaces of V and let X= x + T and Y = y+ T be flats in V.
1) The following are equivalent:
   a) some translate of X is in Y: w + **X** ⊆ Y for some w ϵ V
   b) some translate of S is in T: w + S ⊆ T for some $v \epsilon V$
   c) S ⊆ T
2) The following are equivalent:
   a) X and Y are translates: $w + \boldsymbol{X} = Y$ for some $w \epsilon V$
   b) S and T are translates: $v + S = T$ for some $w \epsilon V$
   c) S=T
3) X ∩ Y≠ **0**, S ⊆ T if and only if **X** ⊆ Y
4) X ∩ Y≠ **0**, S = T if and only if X = Y

**Affine Combinations**
Let X be a nonempty subset of V. It is well known that
1) X is a subspace of V if and only if X is closed under linear combinations, or equivalently, X is closed under linear combinations of any two vectors in X.
2) The smallest subspace of V containing X is the set of all linear combinations of elements of X. In different language, the linear hull of X is equal to the linear span of X.

We wish to establish the corresponding properties of affine subspaces of V, beginning with the counterpart of a linear combination.

**Definition 10**
Let V be a vector space and let xi ϵ V.
A linear combination r1x1 + r2x2 + ………+rnxn where $r_i$ ϵ F and $\sum r_i = 1$ is called an affine combination of the vectors $x_i$.
Let us refer to a nonempty subset X of V as affine closed if X is closed under any affine combination of vectors in X and two- affine closed if X is closed under affine combinations of any two vectors in . These are not standard terms.The line containing two distinct vectors x ,y ϵ V is the set $xy = \{ rx + (1-r)y / r \epsilon F\} = y +$ <x - y>
of all affine combinations of x and y. Thus, a subset X of V is two-affine closed if and only if X contains the line through any two of its points.
**Projective Geometry**

If $dim(V) = 2$, the join (affine hull) of any two distinct points in V is a line. On the other hand, it is not the case that the intersection of any two lines is a point, since the lines may be parallel. Thus, there is a certain asymmetry between the concepts of points and lines in V. This asymmetry can be removed by constructing the projective plane. Our plan here is to very briefly describe one possible construction of projective geometries of all dimensions. By way of motivation, let us consider Figure 1

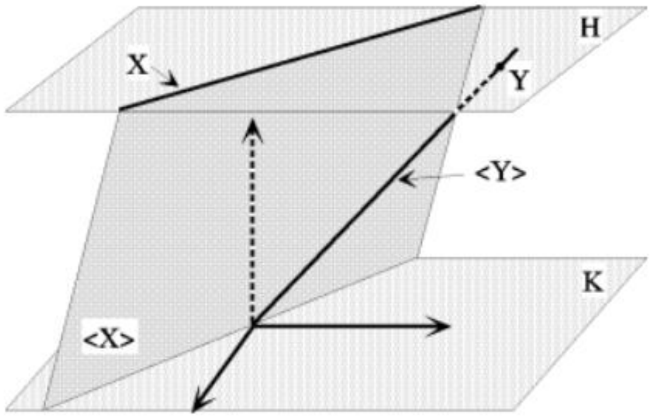


Figure 1

Note that H is a hyper plane in a 3-dimensional vector space V and that 0 does not belongs to H . Now, the set A(H) of all flats of V that lies in H is an affine geometry of dimension 2. (According to our definition of affine geometry, H must be a vector space in order to define A(H). However, we hereby extend the definition of affine geometry to include the collection of all flats contained in a flat of V )The above figure shows a one-dimensional flat X and its linear span < $\boldsymbol{X}$ >as well as a zero-dimensional flat Y and its span < $Y$ >. Note that, for any flat X in H , we have $dim(<\boldsymbol{X}>) = dim(\boldsymbol{X}) + 1$ .Note also that $L_1$ and $L_2$ are any two distinct lines in H, the corresponding
planes and have the property that their intersection is a line through the origin, even if the lines are parallel. We now ready to define projective geometries. Let V be a vector space of any dimension and let H be a hyper plane in V not containing the origin. To each flat in X in H , we associate the subspace of <$\boldsymbol{X}$ > of V generated by X. Thus, the linear span function P: A(H)→S(V) maps affine subspaces X of H to subspaces < $\boldsymbol{X}$ > of V . The span function is not surjective: Its image is the set of all subspaces that are contained in the base subspace K of the flat H.The linear span function is one-to-one and its inverse is intersection with H, P-1U=U∩ $H$ for any subspace not contained in K.The affine geometry A(H) is, as we have remarked, somewhat incomplete. In the case dim(H)=2 , every pair of points determines a line but not every pair of lines determines a point. Now, since the linear span function P is injective, we can identify A(H) with its image P(A(H)) , which is the set of all subspaces of V not contained in the base subspace K . This view of A(H) allows us to "complete" A(H) by including the base subspace K . In the three-dimensional case of above Figure, the 2 base plane, in effect, adds a projective line at infinity. With this inclusion, every pair of lines intersects, parallel lines intersecting at a point on the line at infinity. This two-dimensional projective geometry is called the projective plane

## III. CONCLUSION

Affine spaces became central to physics, computer science, and engineering, especially in the study of kinematics, computer graphics, and machine learning. Concepts such as affine connections (important in general relativity and differen Affine spaces played a role in the broader shift in mathematics toward abstraction and structural approaches in the late 19th and early 20th centuries, exemplified by the works of David Hilbert and Nicolas Bourbaki.tial geometry) and affine manifolds emerged, extending the idea of affine spaces to more complex structures. Affine spaces serve as a bridge between the intuitive geometry of the ancient world and the abstract, algebraic frameworks of modern mathematics. By generalizing Euclidean spaces without the need for a fixed origin, affine spaces capture essential geometric properties such as parallelism and proportionality, providing a versatile foundation for numerous mathematical and practical applications. Their development, rooted in the evolution of geometry from the work of Euclid to the innovations of modern algebra, reflects the progression of mathematical thought toward abstraction and generality. Affine spaces are not only a fundamental concept in geometry but also a critical tool in fields such as physics, computer graphics, optimization, and machine learning. As research continues to uncover new applications and connections, affine spaces remain an indispensable structure in both theoretical exploration and real-world problem-solving. Their timeless relevance underscores the enduring power of mathematical abstraction in shaping our understanding of the universe.

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