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A Note on Pell's Equation

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Abstract
Pell's equation, also called Pell-Fermat equation is a Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a given positive non-square integer and integer solutions are sought for x and y. The aim of this paper is to discuss about some concepts of Pell's equation, examples and also some applications.

Keywords: *Affine space, Convex set.*

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I. INTRODUCTION

Algebraic number theory is a branch of number theory that uses the techniques of abstract algebra to study the integers, rational numbers and their generalization [1]. Pell's equation is an important topic of algebraic number theory that involves quadratic forms and the structure of rings of integers in algebraic number fields. The history of this equation is long and involved a number of different approaches before a definite theory was found. There were partial patterns and quite effective methods of finding solutions, but a complete theory did not emerge untill the end of the 18th century.

Pell's equation, also called Pell-Fermat equation [2] is a Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a given positive non-square integer and integer solutions are sought for x and y. In Cartesian coordinates, the equation is represented by a hyperbola; solutions occur whenever the curve passes through a point with integer x and y coordinates, such as the trivial solution with x = 1 and y = 0. Joseph Louis Lagrange [3] demonstrated that, as long as n is not a perfect square, Pell's equation has an endless number of different integer solutions. These solutions can be used to precisely approximate the square root of n using rational values of the type x/y [4].

The history of Pell's equation is a fascinating journey across different cultures and centuries, highlighting contributions from Indian, Islamic, and European mathematicians [5]. Despite its association with John Pell, the equation predates him by many centuries. It was first studied extensively in India starting with Brahmagupta who found an integer solution to

$$92x^2 + 1 = y^2$$

Then Baskara II who developed a method called "Chakravala method" to solve this equation and Narayana who found the solutions in certain difficult cases [6]. The contributions of the "Greek and Hindu Mathematics" to Pell's equation was aptly brought out by a popular work of B L Vander Waerden [7]. Fermat [8] was also interested in the Pell's equation and worked out some of the basic theories regarding Pell's equation. It was Lagrange who discovered the complete theory of the equation $x^2 - dy^2 = 1$. The name of Pell's equation arose from Leonhard Euler mistakenly attributing Brouckers solution of the equation to John Pell. Mathematicians in the Islamic Golden Age, such as Al-Khwarizmi [9,10], contributed to algebra and number theory, but Pell's equation was not a central focus in Islamic mathematics. However, their work helped transmit mathematical ideas, including methods that were precursors to solving quadratic forms. Fermat rediscovered the equation in the 17th century while studying Diophantine equations. He challenged contemporaries to solve specific cases, such as $x^2-61y^2=1x^2-61y^2=1$, which he claimed was difficult but solvable. Wallis [11,12] included Pell's name in the context of quadratic equations, mistakenly attributing significant work to him. Pell himself had no notable contributions to this equation, but the misattribution stuck, and the equation became known as "Pell's equation." Euler [13,14] made significant strides in formalizing solutions to Pell's equation using continued fractions. He acknowledged the earlier work of Indian mathematicians but developed the methods within the European mathematical framework. Lagrange provided a complete proof that continued fractions can solve Pell's equation. His work made the connection between number theory and Pell's equation systematic and rigorous. Pell's equation remains a central topic in number theory, influencing areas like continued fractions, algebraic number theory, and cryptography [15,16]. It exemplifies how ancient mathematical problems have evolved and interconnected over time.

II. DEFINITION AND PROPERTIES

Definition 1
A Diophantine equation is a polynomial equation usually involving two or more unknowns such that the only solutions of interest are the integer ones. A linear Diophantine equation of the form $ax + by = c$, where a, b, c are integers and has the solution if and only if $(a, b)|c$.

Definition 2
If a and b are integers and there is some integer c such that $a = bc$, then we say that b divides a or b is a factor or a divisor of a and denote $b|a$.

Theorem 1
Every composite numbers greater than one can be expressed as a product of primes and this factorization is unique. It is also known as Unique Factorization Theorem or the Unique Prime Factorization.

Definition 3
Given two integers a and b with $b > 0$ there exist unique integers q and r satisfying $a = qb + r$ The integers q and r are called quotient and remainder respectively in the division of a by b .

Definition 4
The greatest common divisor (m, n) of integers m and n is the largest integer which divides both m and n.
If m and n are relatively prime then $(m, n) = 1$

Lemma 1
If prime p divides the product ab of two integers a and b, then p must divide at least one of those integers a and b

Definition 5
For a positive d integer that is not a perfect square, an equation of the form $x^2 - dy^2 = 1$is called Pell's equation. We are interested in x and y that are both integers, and the term "solution" will always mean an integral solution. The obvious solutions $(x, y) = (\pm1, 0)$, are called the trivial solutions. They are the onlys olutions where $x = \pm1$ or $y = 0$ (separately). Solutions where $x > 0$ and $y > 0$ will be called positive solutions. Every non-trivial solution can be made into a positive solution by changing the sign of x or y.

We don't consider the case when d is a square, since if $d = c^2$ with $c \in Z$ then $x^2 - dy^2 = x^2 - (cy)^2$ and the only squares that differ by 1 are 0 and 1, so $x^2 - (cy)^2 = 1 \Longrightarrow x = \pm1$ and $y = 0$. Thus Pell's equation for square d only has trivial solutions. In this article, we'll show how solutions to Pell's equation can be found, we'll discuss an elementary problem about polygonal numbers that is equivalent to a specific Pell's equation, describes how to create new solutions of Pell's equation if we know one non-trivial solution and we will see how all solutions can be generated from a minimal non-trivial solution. Also we discuss a generalized Pell's equation is introduced, where the right side is not 1.

Example 1
Two positive solutions of $x^2 - 2y^2 = 1$ are (3, 2) and (17, 12),since $2y^2 + 1$ is a square when y = 2 and 12, where it has values 9 = 32 and 289 = 172.Here let (x, y) be the solution. Now we write the Pell's equation as $x^2 = dy^2 + 1$. Here d = 2, so the equation becomes $x^2 = 2y^2 + 1$. Now we give values for y.Take y = 1, 2, 3, ...When y = 1 then the equation becomes $x^2 = 3$ which is not a perfect square. When y = 2 then the equation becomes $x^2 = 9$ which is a perfect square. Then x = 3. Therefore the solution is (3, 2) Now to find the other solutions take number $\alpha_n = (x+y\sqrt{d})^n$ where n = 2, 3, 4, ... Then if we take different values for n, then we get the solution as the coefficients of x.

$$\alpha = 3 + 2\sqrt{2}, \qquad d = 2$$
$$\alpha^n = (3 + 2\sqrt{2})^n$$
$$\alpha^2 = (3 + 2\sqrt{2})(3 + 2\sqrt{2}), \qquad n = 2$$
$$= 9 + 3 \cdot 2\sqrt{2} + 3 \cdot 2\sqrt{2} + 8$$
$$= 17 + 12\sqrt{2}$$

∴(17, 12) is the other solution.

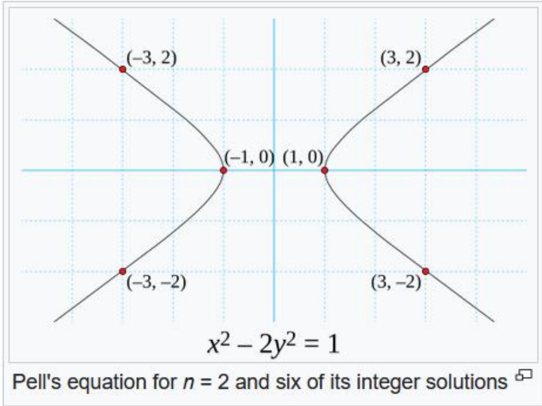

Figure 1: Solution of Pell's Equation

Theorem 2
For all d ∈ Z + that are not squares, the equation has a non-trivial solution.

Proof.
This theorem is our hunting license to search for solutions by tabulating dy^2+1 until it takes a square value. We are guaranteed this search will eventually terminate, but we are not assured how long it will take. In fact, the smallest positive solution of $x^2 - dy^2 = 1$ can be unusually large compared to the size of d. The table illustrates this if we compare the smallest positive solution were d = 12, 13 and 14. As more extreme examples, see the smallest positive solutions below when d = 61 or 109 compared with nearby values of d.

While Lagrange was the first person to give a proof that Pell's equation for general (non-square) has a non-trivial solution. 100 years earlier Fermat claimed to have a proof and challenged other mathematicians in Europe to prove it. In one letter he wrote that anyone failing this task should at least try to find solutions to $x^2 - 61y = 12$ and $x^2 - 109y^2 = 1$, where he said he choose small coefficients "pour ne vous donner pas trop de peine" (so you don't have too much work). He clearly was being mischievous. If Fermat had posed his challenge to mathematicians in India then he may have gotten a positive response; a non-trivial solution to $x^2 - 61y^2 = 1$ had already been known there for 500 years.

Triangular Square Numbers

A positive integer n is called triangular if n dots can be arranged to look like an equilateral triangle. The first six triangular numbers are 1 (a generate case),3,6,10,15 and 21. In the pictures below, the shading shows how each triangular number is built from the previous one by adding a new side.

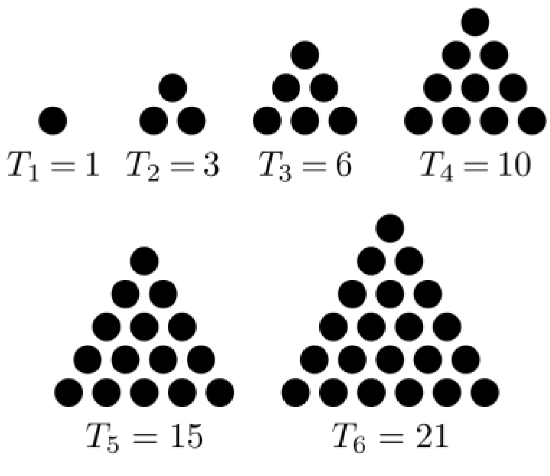

Figure 2: Triangular Square Numbers

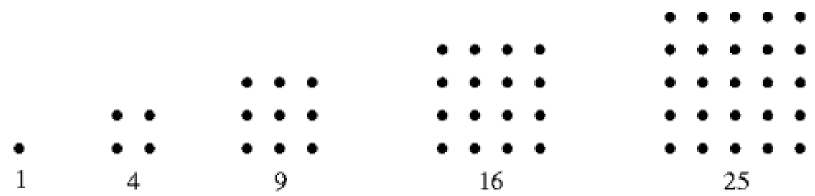

Figure 3: Rectangular Square Numbers

For $k \geq 3$, a k-gonal number is a positive integer n for which n dots can be arranged to look like a regular k-gon. The first six square and pentagonal numbers, corresponding to k = 4 and k = 25, are shown below. Both sequence start with 1 as a generate case.

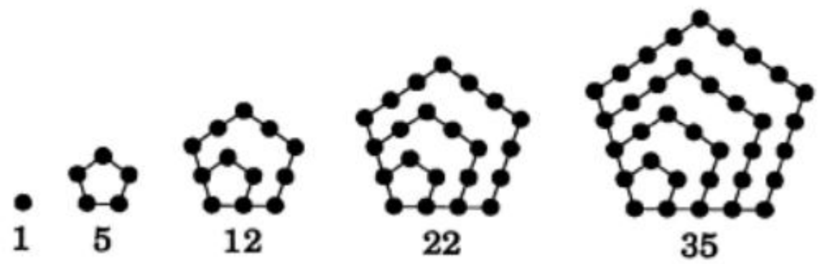

Figure 4: Pentagonal Square Numbers

A formula for the n th square number Sn is obvious. Sn = n 2. To get the formula for the nth triangular and pentagonal numbers Tn and Pn, the few values suggest how to write them as a sum of terms in an arithmetic progression (which are their real definitions).

$$T_n = 1 + 2 + 3 + ... + n =\sum_{k=1}^{n} k$$

$$P_n = 1 + 4 + ... + (3n - 2) = \sum_{k=1}^{n}(3k - 2)$$

This works for square numbers too.

$$S_n = 1 + 3 + ... + (2n - 1) = \sum_{k=1}^{n}(2k - 1)$$

The m^{th} triangular number is m(m+1)/2 and the n^{th} square number is n^2. Using the formula for the sum of terms in an arithmetic progression,

$$T_n =n(n + 1)/2$$

and

$$P_n =n(3n - 1)/2$$

With these formulas we fill the table below of the first 10 triangular, square and pentagonal numbers.

Besides the common value 1, we see 36 is both triangular and square: $36 = T_8 = S_6$. Call a positive integer a triangular square number if it is both T_m for some m and S_n for some n. Finding these numbers is the same as solving a particular Pell's equation.

Theorem 3
Triangular square numbers correspond to solutions of $x^2 - 2y^2 = 1$ in positive integers x and y.
Proof.
Using the formulas for T_m and S_n,

$$T_m = S_n \Longleftrightarrow m(m + 1)/2= n2$$
$$\Longleftrightarrow m^2 + m = 2n^2$$
$$\Longleftrightarrow (m +12)^2 -14= 2n^2$$

$$\Longleftrightarrow (2m+1)^2 - 1 = 2(2n)^2$$
$$\Longleftrightarrow (2m+1)^2 - 2(2n)^2 = 1$$

Because every step is reversible, finding triangular square numbers is equivalent to solving $x^2 - 2y^2 = 1$ in positive integer x and y where $x = 2m + 1$ is odd and $y = 2n$ is even $T_{x-1/2} = S_{y/2}$. (While we want $x = 2m + 1$ with $m \geq 1$, we can say $x > 0$ instead of $x \geq 3$, because the only solution of $x^2 - 2y^2 = 1$ with $x = 1$ has $y = 0$, which is not positive).

Including the constraint that x is odd and y is even in the correspondence between triangular square numbers and positive solutions of $x^2 - 2y^2 = 1$ is unnecessary because they are forced by the equation $x^2 - 2y^2 = 1$. Indeed, writing theequation as $x^2 = 2y^2 + 1$ shows x^2 is odd. Then $x = 2m + 1$ for some integer m, and feeding that into the Pell's equation makes $4m^2 + 4m + 1 - 2y^2 = 1$, so $y^2 = 2m^2 + m$. Thus y^2 is even, so y is even.

III. CONCLUSION

Our aim was to take a note on Pell's Equation by its breath of coverage. As part of algebraic number theory, Pell's equation has applications to computer science, factoring of large integers and cryptography. It is important for certain areas of research in cryptography, although I do not know of any current implementations of it that are actually used in practice. Pell's equation was primarily an exercise in pure mathematics and developing number theory. However, the full importance of it in those fields would not become apparent until the development of algebraic number fields. This project work helps us to know more about Pell's Equation and its application.

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