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Some Integrals Involving H-function Of One – Variable

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Abstract :-
This paper contains ten integrals involving H-function of one variable with proper conditions of validity. The special cases for G-functions have also been obtained.
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1.1 Introduction & Preliminaries

Euler's gamma function (1729), is denoted and defined as follows :

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \text{ provided } Re(x) > 0 \qquad \ldots(1.1.1)$$

The Contour integral form of $_2F_1$ [z] is as follows :

$$2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\middle| z\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)(-z)^s}{\Gamma(c+s)} ds \qquad \ldots(1.1.2)$$

The path L is a contour runs from –i∞ to + i∞. [1, p.49]. The contour integral form of $_pF_q$ is as follows:

$$_pF_q\left[\begin{matrix} a_1......, a_p \\ b_1......, b_q \end{matrix} \middle| z\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma(a_j+s)\Gamma(-s)(-z)^s \prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{q}\Gamma(b_j+s)\prod_{j=1}^{p}\Gamma(a_j)} ds \qquad \ldots(1.1.3)$$

L is a contour runs from – i∞ to + i∞, no a_j (j = 1,…., p) is zero or a negative integer.

If p = q + 1, RHS of above equation is an analytic function of z in the cut plane |arg (–z)| <π.

If p = q, RHS of above equation is an analytic function of z in the open half plane |arg (–z)| <1/2 π i.e. in Re(z) < 0

The contour integral form of Wright's generalized hyper geometric function is defined as follows:-

$$_p\Psi_q(z) = \frac{1}{2\pi i}\int_D \frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j s)\Gamma(-s)(-z)^s}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j s)} ds \qquad \ldots(1.1.4),$$

where D is a contour in the complex s-plane which runs from s =σ – i∞ to s =σ+ i∞ (σ is an arbitrary real number) so that the points s = 0,1,2, … resp. lie to the right of D.

${}_p\psi_q$ makes sense and defines an analytic function of z if

(i) $\mu > 0$ and $z \neq 0$

(ii) $\mu = 0$, $0 < |z| < \xi^{-1}$

If $\mu = 0$, then ${}_p\psi_q(z)$ can be continued analytically into the sector $|\arg(-z)| < \pi$ by means of (1.1.4). …(1.1.5)

The E-function was further generalized by C.S. Meijer (1941, 46) [7,8] which became popular as G-function. It is defined in the following manner :

$$G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} x^s ds \quad \ldots(1.1.6)$$

where an empty product is interpreted as unity; $0 \le m \le q$, $0 \le n \le p$ and parameters are so chosen that no pole of $\Gamma(b_j - s)$, j = 1,2, …., m coincides with any pole of $\Gamma(1 - a_j + s)$, j = 1,2, …., n.

There are three different paths 'L' of integration in (1.1.6) ;

(1) the contour 'L' runs from $-i\infty$ to $+i\infty$ so that all the poles of $\Gamma(b_j - s)$; j = 1 (1) m are to the right and all the poles of $\Gamma(1 - a_j + s)$; j = 1 (1) n to the left of 'L'. The integral converges if $p + q < 2(m + n)$ and $|\arg(x)| < (m + n - p/2 - q/2)\pi$.

(2) the contour 'L' is a loop starting and ending at $+\infty$ and encircling all the poles of $\Gamma(b_j - s)$; j = 1(1) m once in the negative direction but none of the poles of $\Gamma(1 - a_j + s)$; j = 1(1) n. The integral converges if $q \ge 1$ and either $p < q$ or $p = q$ & $|x| < 1$.

(3) the contour 'L' is a loop starting and ending at $-\infty$ and encircling all the poles of $\Gamma(1 - a_j + s$; j = 1(1) n once in the positive direction but none of the poles of) $\Gamma(b_j - s)$; j = 1(1) m. The integral converges if $p \ge 1$ and either $p > q$ or $p = q$ and $|x| > 1$.

The G-function is an analytic function of x, it is symmetric in the parameters a_1 …., a_n likewise in a_{n+1} …. a_p, in b_1…., b_m and b_{m+1}, …., b_q.

The H-function is defined and represented by means of the following Mellin-Barnes type of contour integrals:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \psi(s) x^s ds \quad \ldots(1.1.7),$$

where

(a) $$\psi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)} \quad \ldots(1.1.8),$$

(b) m, n, p and q are non negative integers satisfying $0 \le n \le p$, $0 \le m \le q$

(c) $i = \sqrt{-1}$, $x \neq 0$ is a complex variable.

(d) L is a straight line parallel to the imaginary axis which runs from $-i\infty + i\infty$ with indentations, if necessary, to ensure that the points

$$s = \frac{a_j - \lambda - 1}{\alpha_j}; \; (j = 1, \ldots, n; \lambda = 0,1,2,\ldots)$$

which are the poles of $\Gamma(1 - a_j + \alpha_j s) \forall$ j = 1, …, n lie to the left of L and the points

$$s = \frac{b_h + \upsilon}{\beta_h}; (h = 1, ..., m; \upsilon = 0, 1, 2, ...)$$

which are the poles of of $\Gamma(b_j - \beta_j s) \forall$ j = 1, …, m lie to the right of L. Therefore $\alpha_j(b_h + \upsilon) \neq \beta_h (a_j - \lambda - 1)$; j = 1, …, n; h = 1, …, m; $\lambda, \upsilon = 0,1,2….$ …(1.1.9)

The contour 'L' exists on account of (1.1.9).

(e) a_j (j = 1, …, p) and b_j (j = 1, …, q) are complex numbers and the parameters α_j (j = 1, …, p) and β_j (j = 1, …, q) are positive real numbers.

(f) $\left((a_p, \alpha_p)\right) \equiv (a_1, \alpha_1), (a_2, \alpha_2), ..., (a_p, \alpha_p) \equiv (a_j, \alpha_j)_{1,p}$

(g) The H-function is an analytic function of x if the following conditions are true :

$$\left.\begin{array}{ll} \text{(i)} & x \neq 0, \xi > 0, |\arg x| \leq \tfrac{1}{2}\pi \\ & \text{where } \xi = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \\ \text{(ii)} & \xi = 0 \text{ and } 0 < |x| < \mu^{-1} \\ & \text{where } \mu = \prod_{j=1}^{p} (\alpha_j)^{\alpha_j} \prod_{j=1}^{q} (\beta_j)^{-\beta_j} \end{array}\right] \quad \text{…(1.1.10)}$$

Due to the occurrence of the factor x^s in the integrand of (1.1.7) it is, in general, multiple-valued, but it is one-valued on the Riemann surface of log x.

The H-function of one variable defined by (1.1.7) will be denoted by symbol H(x) finally, for the sake of brevity, the H-function of the form

$$H_{p+1,\,q+1}^{l,\,m+1}\left[z\left|\begin{matrix}(\mu,\lambda),((a_p,A_p))\\((b_q,B_q)),(\gamma,\delta)\end{matrix}\right.\right] \text{ will be abbreviated as } H_{p+1,\,q+1}^{l,\,m+1}\left[z\left|\begin{matrix}(\mu,\lambda),((\quad))\\((\quad)),(\gamma,\delta)\end{matrix}\right.\right]$$

H-function of the form $H_{p,\,q}^{l,\,m}\left[ze^{-\lambda t}\left|\begin{matrix}((a_p,A_p))\\((b_q,B_q))\end{matrix}\right.\right]$ will be abbreviated as $H\left[ze^{-\lambda t}\right]$.

Gottlieb [6] introduced a function called Gottlieb polynomial which is defined & denoted in the following manner:

$$\varphi_n(x;\lambda) = e^{-n\lambda}\, {}_2F_1\left[\begin{matrix}-n, -x\\1\end{matrix}\left|1 - e^{\lambda}\right.\right]$$

1.2 Some Known Results

The following known results will be required in the proof of the integrals (involving H-function of one variable) to be evaluated.

(i) $$\int_0^\infty e^{-st}\varphi_n(x;-t)\,dt = \frac{\Gamma(s-n)\Gamma(s+x+1)}{\Gamma(s+1)\Gamma(s+x-n+1)}$$ provided Re(s) > 0. [9, p.303] …(1.2.1)

(ii) $$\int_{-1}^{1}(1+x)^{\lambda-1}p_\upsilon(x)\,dx = \frac{2^\lambda\left[\Gamma(\lambda)\right]^2}{\Gamma(\lambda+\upsilon+1)\Gamma(\lambda-\upsilon)}$$ provided Re(λ) > 0. [3, p.316(15)] …(1.2.2)

(iii) $\int_0^{\pi} (\sin t)^{\alpha} e^{i\beta t} dt = \frac{\pi}{2^{\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma\left(1+\frac{\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\alpha-\beta}{2}\right)} e^{i\frac{\pi}{2}\beta}$ provided Re(α) > - 1. [1, p.12(29)] …(1.2.3)

(iv) $\int_0^{\pi} (\cos t)^{\alpha} \cos(\beta t) dt = \frac{\pi\,\Gamma(1+\alpha)}{2^{\alpha+1}\Gamma\left(1+\frac{\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\alpha-\beta}{2}\right)}$ provided Re(α) > -1.[1, p.12(30)] …(1.2.4)

1.3 Single Integral Involving H-function Of One Variable

The integrals to be evaluated here are expressed in the form of the following theorems :

Theorem (1.3.1)

If the following sets of conditions are satisfied:

(i) $\lambda > 0, \mathrm{Re}(\sigma) + \lambda_{1\le j\le l}^{\min} \mathrm{Re}\left(\frac{bj}{Bj}\right) > 0$ and

(ii) The H-function of one variable occurring in (1.3.1) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\infty} e^{-\alpha t} \varphi_n(x;-t) H\left[z e^{-\lambda t}\right] dt = H_{p+2,\,q+2}^{l,\,m+2}\left[z \left| \begin{matrix} (1+n-\sigma,\lambda), (-x-\sigma,\lambda), ((\quad)) \\ ((\quad)), (-\sigma,\lambda), (n-x-\sigma,\lambda) \end{matrix} \right.\right] \quad \ldots(1.3.1)$$

Proof : Expressing H-function in the left hand side, of (1.3.1) in contour integral form by (1.1.7); changing the order of t-integral and contour integral,

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - B_j s) \prod_{j=1}^{m} \Gamma(1 - a_j + A_j s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + B_j s) \prod_{j=m+1}^{p} \Gamma(a_j - A_j s)} z^s \cdot \int_0^{\infty} \varphi_n(x;-t) e^{-\lambda s t - \sigma t} dt\, ds$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - B_j s) \prod_{j=1}^{m} \Gamma(1 - a_j + A_j s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + B_j s) \prod_{j=m+1}^{p} \Gamma(a_j - A_j s)} z^s \cdot \left\{\int_0^{\infty} e^{-(\sigma+\lambda s)t} \varphi_n(x;-t) dt\right\} ds$$

on integrating the inner t-integral with the help of (1.2.1) we get

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - B_j s) \prod_{j=1}^{m} \Gamma(1 - a_j + A_j s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + B_j s) \prod_{j=m+1}^{p} \Gamma(a_j - A_j s)} \cdot \frac{\Gamma(\sigma - n + \lambda s)\Gamma(\sigma + x + 1 + \lambda s)}{\Gamma(1 + \sigma + \lambda s)\Gamma(1 - n + x + \sigma + \lambda s)} z^s ds \quad \ldots(1.3.1.1)$$

Now on interpreting (1.3.1.1) with the help of (1.1.7) the right hand side of (1.3.1) follows immediately.

Theorem (1.3.2)

If the following sets of conditions are satisfied :

(i) $\mu > 0, \operatorname{Re}(\sigma) - \mu_{1\le j\le m}^{\max} \operatorname{Re}\left(\frac{a_j - 1}{A_j}\right) > 0.$ and

(ii) the H-function occurring in (1.3.2) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^\infty e^{-\sigma t}\varphi_n(x;-t)H_{p,q}^{l,m}\left[ze^{\mu t}\left|\begin{matrix}((\ \ ))\\((\ \ ))\end{matrix}\right.\right]dt = H_{p+2,\,q+2}^{l+2,m}\left[z\left|\begin{matrix}((\ \ )),(1+\sigma,\mu),(1-n+x+\sigma,\mu)\\(\sigma-n,\mu),(1+x+\sigma,\mu),((\ \ ))\end{matrix}\right.\right] \quad \ldots(1.3.2)$$

Proof : To prove (1.3.2), expressing H-function of the one variable in the left hand side, of (1.3.2) in contour integral form by (1.1.7);

$$= \int_0^\infty e^{-\sigma t}\varphi_n(x;-t)\cdot\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_1 - B_1 s)\prod_{j=1}^{m}\Gamma(1-a_j+A_j s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=m+1}^{p}\Gamma(a_j - A_j s)} z^s e^{\mu ts} ds\, dt$$

changing the order of t-integral and contour integral,

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j - B_j s)\prod_{j=1}^{m}\Gamma(1-a_j+A_j s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=m+1}^{p}\Gamma(a_j - A_j s)} z^s\left[\int_0^\infty e^{-(\sigma-\mu s)t}\varphi_n(x;-t)dt\right]ds$$

on integrating the inner t-integral with the help of (3.2.1) we get

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j - B_j s)\prod_{j=1}^{m}\Gamma(1-a_j+A_j s)}{\prod_{l+1}^{q}\Gamma(1-b_j+B_j s)\prod_{m+1}^{p}\Gamma(a_j - A_j s)}\cdot\frac{\Gamma(\sigma-n-\mu s)\Gamma(\sigma+x+1-\mu s)}{\Gamma(\sigma+1-\mu s)\Gamma(\sigma+x-n+1-\mu s)} z^s ds \quad \ldots(1.3.2.1)$$

Now on interpreting (1.3.2.1) with the help of (1.1.7) the right hand side of (1.3.2) is obtained.

<u>Theorem (1.3.3)</u>

If the following sets of conditions are satisfied:

(i) $h > 0, \operatorname{Re}(\lambda) > 0, \operatorname{Re}(\lambda) + h_{1\le j\le l}^{\min} \operatorname{Re}\left(\frac{bj}{Bj}\right) > 0$ and

(ii) the H-function of one variable occurring in (1.3.3) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_{-1}^1 (1+x)^{\lambda-1} p_\upsilon(x) H_{p,q}^{l,m}\left[z\left(\frac{1+x}{2}\right)^h\left|\begin{matrix}((\ \ ))\\((\ \ ))\end{matrix}\right.\right]dx = 2^\lambda H_{p+2,\,q+2}^{l,m+2}\left[z\left|\begin{matrix}(-\lambda,h),(-\lambda,h),((\ \ ))\\((\ \ )),(\pm\upsilon-\lambda,h)\end{matrix}\right.\right] \quad \ldots(1.3.3)$$

Proof: To prove (1.3.3), expressing H-function of one variable in the left hand side of (1.3.3) in contour integral form by (1.1.7);

$$= \int_{-1}^1 (1+x)^{\lambda-1} p_\upsilon(x)\cdot\frac{1}{2\pi i}\int_L \psi(s) z^s \frac{(1+x)^{hs}}{2^{hs}} ds\, dx.$$

Where $\psi(s)$ is given by (1.1.8). Changing the order of x-integral and s-integral,

$$= \frac{1}{2\pi i}\int_L \frac{\psi(s) z^s}{2^{hs}}\int_{-1}^1 (1+x)^{\lambda+hs-1} p_\upsilon(x) dx\, ds$$

On integrating the inner integral with the help of (1.2.2) we get

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma\left(b_j - B_j s\right)\prod_{j=1}^{m}\Gamma\left(1-a_j+A_j s\right)2^{\lambda+hs}\left[\Gamma\left(\lambda+hs\right)\right]^2}{\prod_{j=l+1}^{q}\Gamma\left(1-b_j+B_j s\right)\prod_{j=m+1}^{p}\Gamma\left(a_j-A_j s\right)2^{hs}\Gamma\left(\lambda+hs+\upsilon+1\right)\Gamma\left(\lambda+hs-\upsilon\right)} z^s ds \qquad \ldots(1.3.3.1)$$

Now on interpreting (1.3.3.1) with the help of (1.1.7) the right hand side of (1.3.3) follows immediately.

Theorem (1.3.4)

If the following conditions are satisfied:

(i) $k > 0, \operatorname{Re}(\lambda) > 0, \operatorname{Re}(\lambda) - k_{1\le j\le m}^{\max}\left(\frac{a_j - 1}{A_j}\right) > 0$ and

(ii) the H-function of one variable occurring in (1.3.4) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_{-1}^{1}(1+x)^{\lambda-1} p_\upsilon(x) H_{p,q}^{l,m}\left[z\left[\tfrac{1}{2}(1+x)\right]^{-k}\middle|\begin{matrix}((\quad))\\((\quad))\end{matrix}\right]dx = 2^{\lambda} H_{p+2,\,q+2}^{l+2,\,m}\left[z\middle|\begin{matrix}((\quad)),(1+\lambda+\upsilon,k),(\lambda-\upsilon,k)\\(\lambda,k),(\lambda,k),((\quad))\end{matrix}\right] \qquad \ldots(1.3.4)$$

Proof : The proof is similar to that of theorem (1.3.3).

Theorem (1.3.5)

If the following conditions are satisfied:

(i) $\mu > 0, \operatorname{Re}(\sigma) + \mu_{1\le j\le l}^{\min} \operatorname{Re}\left(\frac{b_j}{B_j}\right) > -1$ and

(ii) the H-function occurring in (1.3.5) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_{-1}^{1}(1+x)^{\sigma} p_\upsilon(x) H\left[z(1+x)^{\mu}\right]dx = 2^{\sigma+1} H_{p+2,\,q+2}^{l,\,m+2}\left[2^{\mu} z\middle|\begin{matrix}(-\sigma,\mu),(-\sigma,\mu),((\quad))\\((\quad)),(-1-\sigma-\upsilon,\mu),(\upsilon-\sigma,\mu)\end{matrix}\right] \qquad \ldots(1.3.5)$$

Proof : Proof of this theorem is similar to that of (1.3.4).

Theorem (1.3.6)

If the following conditions are satisfied:

(i) $\operatorname{Re}(\alpha) > -1, \mu > 0, \operatorname{Re}(\alpha) + \mu_{1\le j\le l}^{\min} \operatorname{Re}\left(\frac{b_j}{B_j}\right) > -1$ and

(ii) the H-function of one variable occurring in (1.3.6) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\pi}(\sin t)^{\alpha}e^{i\beta t}H_{p,q}^{l,m}\left[x(2\sin t)^{\mu}\left|\begin{matrix}((\quad))\\((\quad))\end{matrix}\right.\right]dt=2^{-\alpha}\pi e^{i\frac{\pi}{2}\beta}H_{p+1,q+2}^{l,m+1}\left[x\left|\begin{matrix}(-\alpha,\mu),((\quad))\\((\quad)),\left(\frac{-\alpha-\beta}{2},\frac{\mu}{2}\right),\left(\frac{\beta-\alpha}{2},\frac{\mu}{2}\right)\end{matrix}\right.\right]\quad\ldots(1.3.6)$$

Proof : To prove (1.3.6) expressing H-function of one variable in L.H.S of (1.3.6) in contour integral form by (1.1.7) and changing the order of t-integral and contour integral and then on integrating the inner t-integral with the help of (1.2.3), we get

$$=\frac{1}{2\pi i}\int_L\frac{2^{\mu s}\pi e^{i\pi\beta/2}}{2^{\alpha+\mu s}}\cdot\frac{\prod_{j=1}^{l}\Gamma(b_j-B_js)\prod_{j=1}^{m}\Gamma(1-a_j+A_js)\Gamma(1+\alpha+\mu s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=m+1}^{p}\Gamma(a_j-A_js)\Gamma\left(1+\frac{\alpha+\beta+\mu s}{2}\right)\Gamma\left(1+\frac{\alpha-\beta+\mu s}{2}\right)}x^s ds$$

…(1.3.6.1)

on interpreting the contour integral in (1.3.6.1) with the help of (1.1.7),the right hand side of (1.3.6) follows immediately.

Theorem (1.3.7)

If the following conditions are satisfied :

(i) $\zeta>0,\mathrm{Re}(\sigma)>-1,\mathrm{Re}(\sigma)-\zeta\max_{1\le j\le m}\mathrm{Re}\left(\frac{a_j-1}{A_j}\right)>-1$ and

(ii) the H-function of one variable occurring in (1.3.7) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\pi}(\sin t)^{\sigma}e^{i\beta t}H_{p,q}^{l,m}\left[x\left(\tfrac{1}{2}\sin t\right)^{-\zeta}\left|\begin{matrix}((\quad))\\((\quad))\end{matrix}\right.\right]dt$$

$$=2^{-\sigma}\pi\exp\left(i\tfrac{1}{2}\pi\beta\right)H_{p+2,q+1}^{l+1,m}\left[x\left|\begin{matrix}((\quad)),\left(1+\frac{\sigma+\beta}{2},\frac{\zeta}{2}\right),\left(1+\frac{\sigma-\beta}{2},\frac{\zeta}{2}\right)\\(1+\sigma,\zeta),((\quad))\end{matrix}\right.\right]\quad\ldots(1.3.7)$$

Proof : Expressing H-function in the L.H.S of (1.3.7) in contour integral form ; changing the order of t-integral and contour integral and then evaluating the inner integral by using (1.2.3) we obtain,

$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{l}\Gamma(b_j-B_js)\prod_{j=1}^{m}\Gamma(1-a_j+A_js)\Gamma(1+\sigma-\zeta s)x^s\pi e^{\frac{1}{2}\pi\beta i}}{\prod_{j=l+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=m+1}^{p}\Gamma(a_j-A_js)\Gamma\left(1+\frac{\sigma+\beta}{2}-\frac{\zeta}{2}s\right)\Gamma\left(1+\frac{\sigma-\beta}{2}-\frac{\zeta}{2}s\right)2^{\sigma}}ds\quad\ldots(1.3.7.1)$$

on interpreting the contour integral (1.3.7.1) into H-function by using (1.1.7), we get the RHS of (1.3.7).

Theorem (1.3.8) (A)

If the following conditions are satisfied :

(i) $\eta>0,\mathrm{Re}(\lambda)>-1,\mathrm{Re}(\beta)+\eta\min_{1\le j\le l}\mathrm{Re}\left(\frac{b_j}{B_j}\right)>0$

(ii) the H-function occurring in (1.3.8)(A) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\pi}(\sin t)^{\lambda} e^{i\beta t} H_{p,q}^{l,m}\left[xe^{i\eta t} \left| \begin{matrix} ((\quad)) \\ ((\quad)) \end{matrix} \right. \right] dt = 2^{-\lambda}\pi\Gamma(1+\lambda) e^{i\frac{1}{2}\pi\beta} H_{p+1,q+1}^{l,m}\left[xe^{i\pi\frac{1}{2}\eta} \left| \begin{matrix} ((\quad)),\left(1+\frac{\lambda-\beta}{2},\frac{1}{2}\eta\right) \\ ((\quad)),\left(\frac{-\lambda-\beta}{2},\frac{1}{2}\eta\right) \end{matrix} \right. \right]$$

…(1.3.8)(A)

<u>Theorem (1.3.8) (B)</u>

If the following conditions are satisfied :

(i) $\eta > 0, \mathrm{Re}(\lambda) > -1, \mathrm{Re}(\beta) - \eta_{1\le j\le m}^{\max} \mathrm{Re}\left(\frac{a_j - 1}{A_j}\right) > 0$

(ii) the H-function of one variable occurring in (1.3.8)(B) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\pi}(\sin t)^{\lambda} e^{i\beta t} H_{p,q}^{l,m}\left[xe^{-i\eta t} \left| \begin{matrix} ((\quad)) \\ ((\quad)) \end{matrix} \right. \right] dt = \pi 2^{-\lambda}\Gamma(1+\lambda) e^{i\frac{1}{2}\pi\beta} H_{p+1,q+1}^{l,m}\left[xe^{-i\frac{1}{2}\pi\eta} \left| \begin{matrix} ((\quad)),\left(1+\frac{\lambda+\beta}{2},\frac{1}{2}\eta\right) \\ ((\quad)),\left(\frac{\beta-\lambda}{2},\frac{1}{2}\eta\right) \end{matrix} \right. \right]$$

…(1.3.8)(B)

Proof: The proof is similar to that of (1.3.7).

<u>Theorem (1.3.9)</u>

If the following conditions are satisfied :

(i) $\upsilon > 0, \mathrm{Re}(\sigma) + (\upsilon)\cdot_{1\le j\le l}^{\min} \mathrm{Re}\left(\frac{b_j}{B_j}\right) > -1$ and $\lambda > 0$,

$\mathrm{Re}(\mu) + (\lambda)_{1\le j\le l}^{\min} \mathrm{Re}\left(\frac{b_j}{B_j}\right) > 0, \upsilon > \lambda$ and

(ii) the H-function of one variable occurring in (1.3.9) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\pi}(\sin t)^{\sigma} e^{i\mu t} H_{p,q}^{l,m}\left[x(2\sin t)^{\upsilon} e^{i\lambda t} \left| \begin{matrix} ((\quad)) \\ ((\quad)) \end{matrix} \right. \right] dt$$

$$= \frac{\pi}{2^{\sigma}} \exp\left(i\tfrac{1}{2}\pi\mu\right) H_{p+1,q+2}^{l,m+1}\left[xe^{i\frac{1}{2}\pi\lambda} \left| \begin{matrix} (-\sigma,\upsilon),((\quad)) \\ ((\quad)),\left(\frac{-\sigma-\mu}{2},\frac{\upsilon+\lambda}{2}\right),\left(\frac{\mu-\sigma}{2},\frac{\upsilon-\lambda}{2}\right) \end{matrix} \right. \right] \quad \ldots(1.3.9)$$

Proof : To prove (1.3.9), expressing H-function of one variable in L.H.S of (1.3.9) in contour integral form by (1.1.7) and changing the order of t-integral and contour integral,

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j - B_j s)\prod_{j=1}^{m}\Gamma(1-a_j + A_j s)}{\prod_{j=m+1}^{p}\Gamma(a_j - A_j s)\prod_{j=l+1}^{q}\Gamma(1-b_j + B_j s)} x^s 2^{\upsilon s}\int_0^{\pi}(\sin t)^{\sigma+\upsilon s} e^{i(\mu+\lambda s)t} dt\, ds$$

on evaluating the inner t-integral by using (1.2.3), we get

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j-B_js)\prod_{j=1}^{m}\Gamma(1-a_j+A_js)\Gamma(1+\sigma+\upsilon s)\,x2^{\upsilon s}\pi e^{i\frac{\pi}{2}(\mu+\lambda s)}}{\prod_{j=m+1}^{p}\Gamma(a_j-A_js)\prod_{j=l+1}^{q}\Gamma(1-b_j+B_js)\Gamma\left(1+\frac{\sigma+\mu+\upsilon s+\lambda s}{2}\right)\Gamma\left(1+\frac{\sigma-\mu+\upsilon s-\lambda s}{2}\right)2^{\sigma+\upsilon s}}ds$$

$$=\frac{\pi}{2^{\sigma}}\cdot\frac{1}{2\pi i}\int_L e^{i\frac{1}{2}\pi\mu}\frac{\prod_{j=1}^{l}\Gamma(b_j-B_js)\prod_{j=1}^{m}\Gamma(1-a_j+A_js)\Gamma(1+\sigma+\upsilon s)\left(xe^{i\frac{1}{2}\pi\lambda}\right)^s ds}{\prod_{j=m+1}^{p}\Gamma(a_j-A_js)\prod_{j=l+1}^{q}\Gamma(1-b_j+B_js)\Gamma\left(1+\frac{\sigma+\mu}{2}+\left\{\frac{\upsilon+\lambda}{2}\right\}s\right)\Gamma\left(1-\left(\frac{\mu-\sigma}{2}\right)+\left\{\frac{\upsilon-\lambda}{2}\right\}s\right)}$$

…(1.3.9.1)

Now on interpreting (1.3.9.1) into H-function with the help of (1.1.7) the RHS of (1.3.9) follows immediately.

Note : If $\lambda > \upsilon$, then (1.3.9.1) can also be written as

$$=\frac{1}{2\pi i}\frac{\pi}{2^{\sigma}}e^{i\frac{1}{2}\pi\mu}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j-B_js)\prod_{j=1}^{m}\Gamma(1-a_j+A_js)\Gamma(1+\sigma+\upsilon s)}{\prod_{l+1}^{q}\Gamma(1-b_j+B_js)\prod_{m+1}^{p}\Gamma(a_j-A_js)\Gamma\left[1+\frac{\sigma+\mu}{2}+\left(\frac{\upsilon+\lambda}{2}\right)s\right]}\cdot\frac{\left(xe^{i\frac{1}{2}\pi\lambda}\right)^s}{\Gamma\left[1+\frac{\sigma-\mu}{2}-\left(\frac{\lambda-\upsilon}{2}\right)s\right]}ds$$

…(1.3.9.2)

Now on interpreting (1.3.9.2) with the help of (1.1.7), we get

$$\int_0^{\pi}(\sin t)^{\sigma}e^{i\mu t}H_{p,q}^{l,m}\left[x(2\sin t)^{\upsilon}e^{i\lambda t}\middle|\begin{matrix}((\ \ ))\\((\ \ ))\end{matrix}\right]dt$$

$$=\frac{\pi}{2^{\sigma}}\exp\left(i\tfrac{1}{2}\pi\mu\right)H_{p+2,q+1}^{l,m+1}\left[xe^{i\frac{1}{2}\pi\lambda}\middle|\begin{matrix}(-\sigma,\upsilon),((\ \ ))\left(\frac{2+\sigma-\mu}{2},\frac{\lambda-\upsilon}{2}\right)\\((\ \ )),\left(\frac{-\sigma-\mu}{2},\frac{\upsilon+\lambda}{2}\right)\end{matrix}\right]\quad\ldots(1.3.9)(B)$$

This completes the proof.

Theorem (1.3.10)

If the following conditions are satisfied :

(i) $\lambda>0, \mathrm{Re}(\sigma)+\lambda\cdot\min_{1\le j\le l}\mathrm{Re}\left(\frac{b_j}{B_j}\right)>-1$ and

(ii) the H-function of one variable occurring in (1.3.10) satisfies the conditions of analyticity similar to (1.1.10) then,

$$\int_0^{\pi}(\cos t)^{\sigma}(\cos\mu t)H_{p,q}^{l,m}\left[z(2\cos t)^{\lambda}\middle|\begin{matrix}((\ \ ))\\((\ \ ))\end{matrix}\right]dt=\frac{\pi}{2^{1+\sigma}}H_{p+1,q+2}^{l,m+1}\left[z\middle|\begin{matrix}(-\sigma,\lambda),((\ \ ))\\((\ \ )),\left(\frac{-\sigma-\mu}{2},\frac{\lambda}{2}\right),\left(\frac{\mu-\sigma}{2},\frac{\lambda}{2}\right)\end{matrix}\right]$$

…(1.3.10)

Proof : To prove (1.3.10), expressing H-function in the L.H.S of (1.3.10) in contour integral form by (1.1.7); changing the order of integrals and on integrating the inner integral with the help of (1.2.4), we get

$$=\frac{1}{2\pi i}\left(\frac{\pi}{2^{\sigma+1}}\right)\int_L \frac{\prod_{j=1}^{l}\Gamma\left(b_j-B_js\right)\prod_{j=1}^{m}\Gamma\left(1-a_j+A_js\right)\Gamma\left(1+\sigma+\lambda s\right)}{\prod_{j=l+1}^{q}\Gamma\left(1-b_j+B_js\right)\prod_{j=m+1}^{p}\Gamma\left(a_j-A_js\right)\Gamma\left(1+\frac{\sigma+\mu+\lambda s}{2}\right)\Gamma\left(1+\frac{\sigma-\mu+\lambda s}{2}\right)} z^s ds$$

…(1.3.10.1)

Now on interpreting (1.3.10.1) into H-function by (1.1.7) the RHS of (1.3.10) follows immediately.

1.4. Special cases of (1.3) for G-functions

Taking Aj, Bj each equal to unity in (1.3.1) to (1.3.10), we get the corresponding integrals for Meijer's G-function. These are given below :

(1) Taking $\lambda = 1$ in (1.3.1), we get

$$\int_0^\infty e^{-\sigma t}\varphi_n\left(x;-t\right)G_{p,q}^{l,m}\left[ze^{-t}\left|\begin{matrix}\left(\left(a_p,1\right)\right)\\ \left(\left(b_q,1\right)\right)\end{matrix}\right.\right]dt = G_{p+2,q+2}^{l,m+2}\left[z\left|\begin{matrix}(1+n-\sigma,1),(-x-\sigma,1),\left(\left(a_p,1\right)\right)\\ \left(\left(b_q,1\right)\right),(-\sigma,1),(n-x-\sigma,1)\end{matrix}\right.\right]$$

(2) Taking $\mu = 1$ in (1.3.2), we get

$$\int_0^\infty e^{-\sigma t}\varphi_n\left(n;-t\right)G_{p,q}^{l,m}\left[ze^{t}\left|\begin{matrix}\left(\left(a_p,1\right)\right)\\ \left(\left(b_q,1\right)\right)\end{matrix}\right.\right]dt = G_{p+2,q+2}^{l+2,m}\left[z\left|\begin{matrix}\left(\left(a_p,1\right)\right),(1+\sigma,1)(1-n+x+\sigma,1)\\ (\sigma-n,1),(1+x+\sigma,1)\left(\left(b_q,1\right)\right)\end{matrix}\right.\right]$$

(3) If we take h = 1 in (1.3.3), we get

$$\int_{-1}^1 (1+x)^{\lambda-1}P_\upsilon(x)\,G_{p,q}^{l,m}\left[z\left(\frac{1+x}{2}\right)\left|\begin{matrix}((a_p))\\ ((b_q))\end{matrix}\right.\right]dx = 2^\lambda\; G_{p+2,q+2}^{l,m+2}\left[z\left|\begin{matrix}(-\lambda),(-\lambda),((a_p))\\ ((b_q)),(\pm\nu-\lambda)\end{matrix}\right.\right]$$

(4) If we put k = 1 in (1.3.4), we get

$$\int_{-1}^1 (1+x)^{\lambda-1}P_\upsilon(x)\,G_{p,q}^{l,m}\left[z\left(\frac{1+x}{2}\right)^{-1}\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right]dx = 2^\lambda\; G_{p+2,q+2}^{l+2,m}\left[z\left|\begin{matrix}a_1,\dots,a_p,(1+\lambda+\upsilon),(\lambda-\upsilon)\\ \lambda,\lambda,b_1,\dots,b_q\end{matrix}\right.\right]$$

(5) If we put $\mu = 1$ in (1.3.5), we get

$$\int_{-1}^1 (1+x)^{\sigma}P_\upsilon(x)\,G\left[z\left(1+x\right)\right]dx = 2^{\sigma+1}\; G_{p+2,q+2}^{l,m+2}\left[2z\left|\begin{matrix}(-\sigma),(-\sigma),((a_p))\\ ((b_q)),(-1-\sigma-\upsilon),(\upsilon-\sigma)\end{matrix}\right.\right]$$

(6) Taking $\eta = 2$ in (1.3.8), we get

(a) $$\int_0^\pi (\sin t)^\lambda e^{i\beta t}\,G_{p,q}^{l,m}\left[e^{2it}\left|\begin{matrix}((a_p))\\ ((b_q))\end{matrix}\right.\right]dt = 2^{-\lambda}\,\pi\Gamma\left(1+\lambda\right)e^{i\frac{1}{2}\pi\beta}G_{p+1,q+1}^{l,m}\left[xe^{i\pi}\left|\begin{matrix}((a_p)),\left(1+\frac{\lambda-\beta}{2}\right)\\ ((b_q)),\left(\frac{-\lambda-\beta}{2}\right)\end{matrix}\right.\right]$$

(b) $$\int_0^{\pi}(\sin t)^{\lambda}e^{i\beta t}\,G_{p,q}^{l,m}\left[xe^{-2it}\left|\begin{matrix}((a_p))\\((b_q))\end{matrix}\right.\right]dt=\pi 2^{-\lambda}\Gamma(1+\lambda)e^{i\frac{1}{2}\pi\beta}G_{p+1,q+1}^{l,m}\left[xe^{-i\pi}\left|\begin{matrix}((a_p)),\left(1+\frac{\lambda+\beta}{2}\right)\\((b_q)),\left(\frac{\beta-\lambda}{2}\right)\end{matrix}\right.\right]$$

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