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Construction of a New Basis Collocation Method for the Solution of Some Volterra Integral Equations of the Second Kind

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Abstract
This paper utilizes Gram-Schmidt orthogonalization process to construct a new orthogonal polynomial which was used as a basis function with collocation method to obtain the solution of Volterra integral equation of the second kind. The approximate solution obtained using the new method shows a high degree of accuracy when compared with the exact solution for the solved examples.

Keywords: orthogonal polynomial; Volterra integral equation of the second kind; basis function

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I. INTRODUCTION

Many problems in mathematical physics and engineering are often reduced to integral equations of the second kind, which are inherently ill-posed problems and small change to the problem can make very large changes to the solution obtained [1;2]. Integral equations of various types appear in many field of sciences. Differential equations with transformed argument or differential equation of neutral type can be transformed to Volterra Fredholm integral equation [3]. There are several method for approximating the solution of Volterral Integral Equations (VIEs), but very few references have been found in technical literature dealing with VIEs. [4] applied Taylor series method for solving some of such equation. [5] proposed a simple efficient direct method for solving the Volterra integral equations. [6] solved VIEs of the first kind by wavelet basis. [7] applied He's homogony perturbation method to solve systems of VIEs of the first kind. [8] produced the approximation solution of the VIEs of the first kind via a recurrence relations. [9] introduced and used a modification of block pulse functions to solve VIEs of the first kind. In the Literature other authors solve VIEs using moving least Squares and Chebyshev polynomials. [10] solved VIE using composite Collocation method.

In this paper the new orthogonal collocation method was used to solve VIE of the form

$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt \qquad 0 \leq x, t \leq 1 \qquad - \qquad (1)$$

Where u(x) is the unknown functions while f(x) and the kernel k(x,t) is known L^2 function. The outline of the paper is as follows. Section 2 is devoted to the construction of the new basis function in section 3, the new method is applied to (1). In section 4, numerical examples and discussion of results are give and finally conclusion is given in section 5.

II. CONSTRUCTION OF THE NEW BASIC FUNCTION

The construction is based on Gram-Schmidt Orthogonalization Process.

Suppose that the orthogonal polynomial f_i(t) valid on the interval [-1, 1] has the leading term t^i , then starting with

$$f_0(t) = 1 \qquad (2.1)$$

The linear polynomial f_1(t), with leading term t, can be written as

$$t + a_{1,0} f_0(t) \qquad (2.2)$$

Since f_1(t) and f_0(t) are orthogonal, then:

$\int_{-1}^{1} w(t) f_0(t) f_1(t) dt = 0$, where w(t) is the weight function

In this paper, $(1-t)^4$ is chosen as the weight function

$$\int_{-1}^{1} w(t) f_0(t) f_1(t) dt = \int_{-1}^{1} t(1-t)^4 f_0(t) dt + a_{1,0} \int_{-1}^{1} (1-t)^4 f_0^2(t) dt \qquad (2.3)$$

Substituting (2.1) and (2.2) in (2.3) gives:

$a_{1,0} = \frac{-\int_{-1}^{1} t(1-t)^4 dt}{\int_{-1}^{1}(1-t)^4 dt} = \frac{2}{3}$

Substitute 2/3 for $a_{1,0}$ in (2.2) to obtain:

f_1 (t) = t + $^2/_3$

Now, the polynomial f_2(t) of degree 2 in t with leading term t^2 is

$$f_2(t) = t^2 + a_{2,0}f_0 + a_{2,1}f_1(t) \qquad (2.4)$$

$a_{2,0} = \frac{-\int_{-1}^{1} t^2(1-t)^4 dt}{\int_{-1}^{1}(1-t)^4 dt} = \frac{-11}{21}$

$a_{2,1} = \frac{-\int_{-1}^{1} t^2(1-t)^4(t+\frac{2}{3})}{\int_{-1}^{1}(1-t)^4(t+\frac{2}{3})^4 dt} = 1$

Substitute $-\frac{11}{21}$ *and* 1 *for* $a_{2,0}$ *and* $a_{2,1}$ *in* (2.4) to obtain:

$f_2 = t^2 + t + \frac{1}{7}$

Using the above procedure, the following new orthogonal polynomials are obtained:

$$f_3(t) = t^3 + \frac{6}{5}t^2 + \frac{1}{5}t - \frac{1}{15}$$
$$f_4(t) = t^4 + \frac{4}{3}t^3 + \frac{2}{11}t^2 - \frac{12}{15}t - \frac{17}{495}$$
$$f_5(t) = t^5 + \frac{10}{7}t^4 + \frac{10}{91}t^3 - \frac{40}{91}t^2 - \frac{95}{100}t + \frac{10}{1001}$$

$$f_6 = t^6 + \frac{3}{2}t^5 - \frac{5}{7}t^3 - \frac{15}{91}t^2 + \frac{9}{182}t + \frac{8}{1001}$$

And so on.

For analytical and numerical work, which is the focus of this paper, it is convenient to use the half interval $0 \le t \le 1$ instead of the full interval -1 $\le t \le 1$ .For this purpose, the new shifted orthogonal polynomials is obtained from the relation

$$p_i(t) = f_i(2t-1)$$
$$p_0 = 1$$
$$p_1 = 2t - 1 + \frac{2}{3}$$
$$= 2t - \frac{1}{3}$$
$$= \frac{1}{3}(6t-1)$$

Using the above process, the first nine new orthogonal polynomials are given below:

$p0 := 1$

$p1 := \frac{1}{3}(6t-1)$

$p2 := \frac{1}{7}(28t^2 - 14t + 1)$

$p3 := \frac{1}{15}(120t^3 - 108t^2 + 24t - 1)$

$p4 := \frac{1}{495}(7920t^4 - 10560t^3 + 4320t^2 - 576t + 16)$

$p5 := \frac{1}{1001}(32032t^5 - 57200t^4 + 35200t^3 - 8800t^2 + 800t - 16)$

$p6 := \frac{1}{1989}(128128t^6 - 288288t^5 + 240240t^4 - 91520t^3 + 15840t^2 - 1056t + 16)$

$p7 := \frac{1}{663}(509184t^7 - 1386112t^6 + 1467648t^5 - 764400t^4 + 203840t^3 - 26208t^2 + 1344t - 16)$

$$p8 := \frac{1}{16124160}\left(16124160\,t^8 - 51597312\,t^7 + 66533376\,t^6 - 44355584\,t^5 + 16307200\,t^4 - 3300200\,t^3 + 361028\,t^2 - 21064\,t + 451\right)$$

III. METHOD OF SOLUTIONS

In this section, the new basis function is applied to equation (1) to give approximation in the form:

u_N (t) = P(t) A, $N \in Z^+$ … …………………………… (2)

Where P(t) is the newly constructed basis function, p_0 = 1, $P_1 = \frac{1}{3}$(6t-1),…….and A = [a_0, a_1, a_2…a_N] are constant to be determined.

Substitute (2) into (1) to give:

g(t) A = S(t)…………………………………………(3)

Where, $g(t) = P(t) + \int_0^x k\,(x,t)u(t)dt$

S(t) = f(x)

We then collocate (3) using the standard collocation points

$X_i = a + \frac{(b-a)i}{N}$ (4)

to give [g (t_i)] $_{(N+1 \times N+1)}$ A = [s (t_i)] $_{(N+1 \times 1)}$ (5)

We then substitute (5) into (2) to obtain the numerical solution

IV. NUMERICAL EXAMPLES

In this section, two examples are solved to test for the accuracy and reliability of the method using our newly constructed orthogonal polynomials. All computations are done using maple 18 run on a pc.

Example1: consider the following Volterra integral equation.

U (x) = 1 – xSin (x) + x Cos (x) + $\int_0^x t\,U(t)dt$

With the exact solution U (x) = Sin (x) + Cos (x)

U_N (x) is computed as follow; applying the newly constructed polynomial as a basis function gives

$U_N(x) =$

$$-\frac{8}{3}x^6 a_4 + \frac{200}{21}x^6 a_5 - \frac{8}{5}x^5 a_3 + \frac{64}{15}x^5 a_4 - x^4 a_2 + \frac{9}{5}x^4 a_3 - \frac{1}{3}x^3 a_1 + \frac{2}{3}x^3 a_2 - \frac{1}{2}x^2 a_0 + \frac{1}{12}x^2 a_1 - \frac{32}{7}a_5 x^7 - \frac{16}{1001}a_5 + \frac{16}{495}a_4 - \frac{1}{15}a_3 + \frac{1}{7}a_2 + a_0 - \frac{1}{6}a_1 + a_1 x + \frac{55}{14}x^2 a_2 - 2x a_2 + \frac{112}{15}x^3 a_3 - \frac{43}{6}x^2 a_3 + \frac{8}{5}x a_3 + \frac{152}{11}x^4 a_4 - \frac{1152}{55}x^3 a_4 + \frac{392}{45}x^2 a_4 - \frac{64}{55}x a_4 + \frac{2272}{91}x^5 a_5 - \frac{5000}{91}x^4 a_5 + \frac{104800}{3003}x^3 a_5 - \frac{1256}{143}x^2 a_5 + \frac{800}{1001}x a_5 = 1 - x\sin(x) + x\cos(x)$$

Selecting (0, 0.2, 0.4, 0.6, 0.8, 1.0) as the collocation points in the above equation and solving the square matrix will give the numerical solution, thus:

$\mathbf{U_{exact}} = \sin(x) + \cos(x)$

$$\mathbf{u_{approx}} = 1.000000000 + 0.9999284857\,x - 0.4991918030\,x^2 - 0.1699389360\,x^3 + 0.04770421310\,x^4 + 0.003271124561\,x^5$$

Table1. Numerical solution,exact solution & error of example1

x	U_{exact}	U_{approx}	Error
0.0	1.0000000000	1.0000000000	0.0000e+00
0.1	1.0948375820	1.0948357950	1.7873e-06

0.2	1.1787359090	1.1787358880	2.2007e-08
0.3	1.2508566960	1.2508572860	5.8912e-07
0.4	1.3104793360	1.3104793380	2.0905e-09
0.5	1.3570081000	1.3570076610	4.3904e-07
0.6	1.3899780880	1.3899780610	2.7216e-08
0.7	1.4090598740	1.4090604610	5.8692e-07
0.8	1.4140628000	1.4140628280	2.7256e-08
0.9	1.4049368780	1.4049350930	1.7850e-06
1.0	1.3817732910	1.3817730850	2.0664e-07

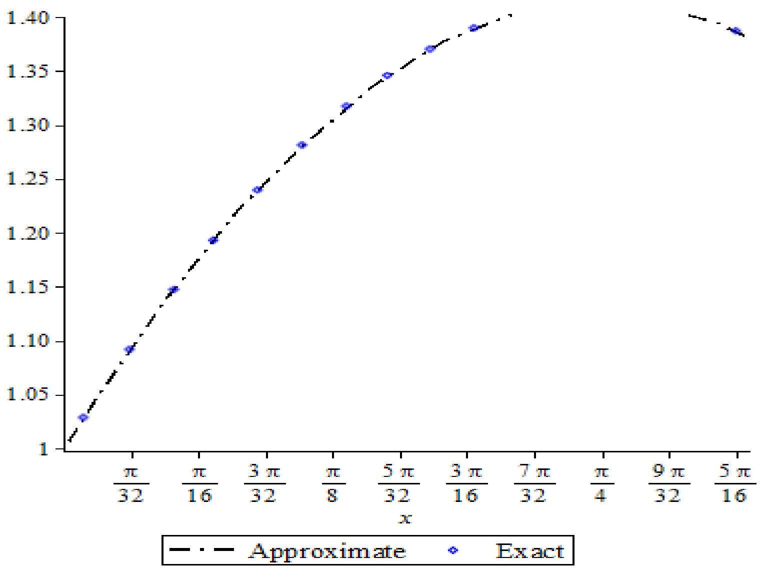

Figure1: Graph of exact solution and numerical solution for example1

4.2 EXAMPLE 2

Consider the following Volterra Integral Equation,

U(x) = $2 + 3x + \frac{3}{2}x^2 + \frac{1}{6}x^3 - \int_0^x (x-t+2)u(t)dt$

with the exact solution U(x) = $1 + e^{-x}$

Applying the method in Example 1 gives:

$$U_x(x)= \frac{728}{165}x^5 a_4 + \frac{1}{15}x^5 a_2 + \frac{17}{25}x^5 a_3 + \frac{43}{15}x^4 a_3 + \frac{1}{24}x^4 a_1 + \frac{7}{12}x^4 a_2 + \frac{85}{42}x^3 a_2$$
$$+ \frac{11}{36}x^3 a_1 + \frac{2}{21}a_5 x^8 + \frac{8}{105}x^7 a_4 + \frac{184}{147}x^7 a_5 + \frac{1952}{273}x^6 a_5 + \frac{1}{15}x^6 a_3$$
$$+ \frac{8}{9}x^6 a_4 + \frac{15}{7}x^2 a_2 - \frac{12}{7}x a_2 + \frac{67}{18}x^3 a_3 - \frac{17}{3}x^2 a_3 + \frac{22}{15}x a_3 + \frac{1112}{165}x^4 a_4$$
$$- \frac{23608}{1485}x^3 a_4 + \frac{752}{99}x^2 a_4 - \frac{544}{495}x a_4 + \frac{488}{39}x^5 a_5 - \frac{123100}{3003}x^4 a_5 + \frac{8072}{273}x^3 a_5$$
$$- \frac{8016}{1001}x^2 a_5 + \frac{768}{1001}x a_5 + x^2 a_0 + 2x a_0 + \frac{5}{6}x^2 a_1 + \frac{2}{3}a_1 x + \frac{1}{6}x^3 a_0 - \frac{16}{1001}a_5$$
$$+ \frac{16}{495}a_4 - \frac{1}{15}a_3 + \frac{1}{7}a_2 + a_0 - \frac{1}{6}a_1 = 2 + 3x + \frac{3}{2}x^2 + \frac{1}{6}x^3$$

Select (0, 0.2π, 0.4π, 0.6π, 0.8π, 1.0π) as collocation points to find the values of the constant terms in the equation above and then substitute those constant terms back into the equations and simplify to obtain our numerical solution;
Thus,

$$\mathbf{u_{exact}} = 1 + e^{-x}$$

$$u_{approx} = 2.000000000 - 0.9968823430x + 0.4859125788x^2 - 0.1447449151x^3 + 0.02547521282x^4 - 0.002007943801x^5$$

Table2: Numerical solution, exact solution & error for example2

x	U_{exact}	U_{approx}	Error
0.0	2.0000000000	2.0000000000	0.0000e+00
0.1	1.9048374180	1.9050286750	1.9126e-04
0.2	1.8187307530	1.8189421920	2.1144e-04
0.3	1.7408182210	1.7409607860	1.4257e-04
0.4	1.6703200460	1.6703610050	4.0959e-05
0.5	1.6065306600	1.6064733120	5.7348e-05
0.6	1.5488116360	1.5486796700	1.3197e-04
0.7	1.4965853040	1.4964111420	1.7416e-04
0.8	1.4493289640	1.4491454630	1.8350e-04
0.9	1.4065696600	1.4064046530	1.6501e-04
1.0	1.3678794410	1.3677525900	1.2685e-04

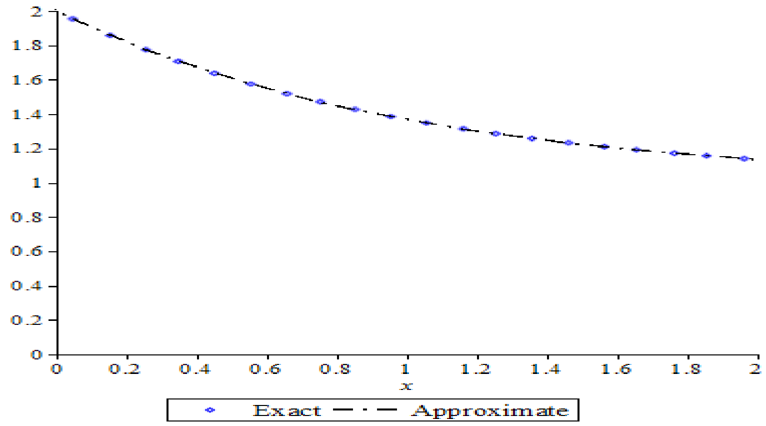

Figure2: Graph of exact solution and numerical solution for example2

V. CONCLUSION

In this paper, a numerical scheme based on the construction of a new basis function in combination with collocation method has been used to obtain approximate solution of the Volterra integral equation of the second kind. From the examples given, the proposed method proved to be very accurate when compared to the exact solutions.

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