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Median Graphs-A Survey

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Abstract
In the following sense, median graphs are a typical generalization of trees and hypercubes: for each three vertices u, v, w there exists a single x that is located on the shortest path connecting any two of u, v, w. Median graphs have their roots in the world of pure mathematics, specifically, semilattices, ternary algebras and Helly hypergraphs. This paper attempts to survey the properties of median graphs.
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I. INTRODUCTION

Median graphs make up a very intriguing class from the perspective of mathematics. A rich structure theory has been created, and there is yet much more to come. An interesting aspect is how median graphs can be seen in various forms in many other branches of mathematics. Equally significant, however, is the fact that median graphs and median-type structures have a wide range of uses in fields as disparate as mathematical biology, psychology, chemistry, economics, and literature.

The classes of trees and hypercubes are two well-known types of graphs. It is interesting to note that that the 0,1-vectors of length n make up the vertices of the n-dimensional hypercube Q^n, often known as the n-cube, and that two vertices are connected by an edge if they differ in precisely one coordinate when they are 0,1-vectors. The ordered graph, obtained by introducing an ordering $\leq$ on the n-cube's vertices, with $u \leq v$ whenever the 0,1-vector of v has a one in every location where the 0,1-vector of u has a one, is the exact Hasse diagram of the n-dimensional Boolean lattice. There are several papers on both groups in the literature. These classes appear to be fairly different at first glance. However, appearances can be deceiving. There is a surprising similarity between trees and hypercubes: given any three vertices, u, v, and w there exists a single vertex x, known as the median of u, v, and w, that simultaneously resides on the shortest path connecting each pair of vertices. Since there is only one path between any two vertices in a tree, this is evident. Each coordinate of x in a hypercube vertex is assigned the value that is most common among all values for the associated coordinates of u, v, and w. The question of which graphs share this trait with trees and hypercubes thus becomes relevant. The median graphs are the graphs in which this characteristic is satisfied. Various authors independently introduced these graphs [1,2]. In Avann's original work on "unique ternary distance graphs.", these graphs essentially received little attention because distributive semi-lattices were the main focus of that work.

In the work by Nebesky,[3] the relation with median algebras was the main point of interest. Last but not least, was the work by Mulder et. al., [5], where the relation with Helly hypergraphs was the focus. The Expansion Theorem, proven in [6], the first study on median graphs to concentrate solely on the graphs (see Section 5.2). Results on the median plots are shown here from a perspective that was developed considerably later. It is based on the notion that median graphs appear as a typical generalization of trees and hypercubes in the above-mentioned theory.

As is demonstrated in the literature, median graphs enable a rich structure theory. There are numerous intriguing generalizations of median graphs, which is also significant. Additionally, median structures are widespread in many other branches of mathematics. Last but not least, median structures and graphs have several uses in fields as diverse as computer science, evolutionary theory, chemistry, literary history, location theory, and consensus theory. With a focus on median graphs, this paper provides an introduction to the theory of median structures and their applications. Metric Graph Theory is a fast-growing subdiscipline of graph theory, with many applications and relations with various other disciplines of mathematics. A prime example, where all these aspects are abundant, is the area of median type problems with the median graphs as landmark. In this paper, the theory of median graphs is surveyed in detail. With some minor adaptations the terminology of Bondy and Murty [4] is adopted.

II. DEFINITIONS AND PRELIMINARIES

The following definitions are as in [7]. In this section, we give the basic definitions and preliminaries of median graphs. A basic property of convex sets is that the family of convex sets is closed under taking arbitrary intersections. Loosely speaking, this is the defining property of a *convexity* in abstract convexity theory. In the infinite case a second defining property is that the union of a nested family of convex sets is again convex. In the sequel we will not distinguish between a subset W of V and the subgraph of G induced by W. The notions of interval and convexity in a graph probably are already part of folklore for a couple of decades. Clearly, modular graphs are bipartite. Also, a convex subgraph of a median graph is again a median graph. Notation and terminology were fixed in [7], where they were studied systematically for the first time.

- **u-v geodesic**: Let $G = (V, E)$ be a finite, connected, simple, loopless graph. For any two vertices *u* and *v* of *G*, the *distance d* (*u*, *v*) between u and v is the length of a shortest *u*-*v*-path, or a *u-v-geodesic*.

- The **interval** between *u* and *v* is the set $I\ (u, v) = \{w|\ d(u, w) + d(w, v) = d(u, v)\}$.
  It consists of all vertices between *u* and *v*.

- A subset W of V is **convex** if $I(x, y)$ is contained in W, for any x, y in W.

- A subgraph H of G is **convex** if it is induced by a convex set in G.

- For a set $W \subseteq V$, the **convex closure Con[W]** is the smallest convex set containing *W*.

- For u, v, w in G, we write $\boldsymbol{I(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, v)}$**.** An example is given by Fig:1

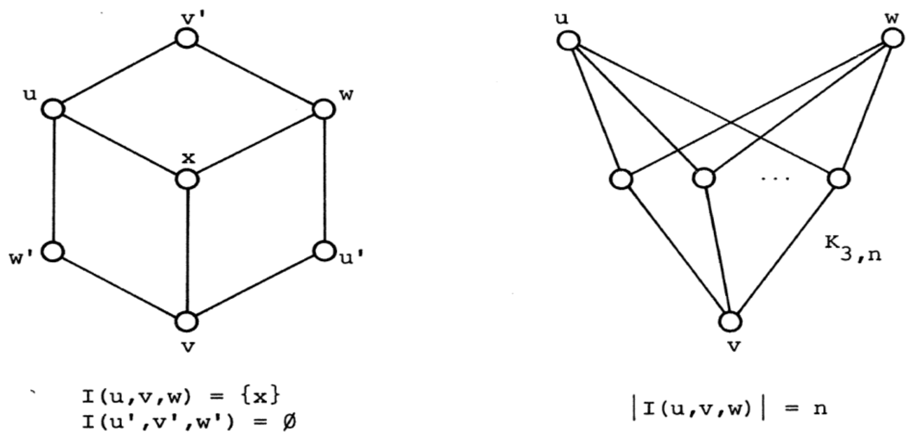

Fig: 1

- If $I(u, v, w) \neq \varphi$ or all triples u, v, w in G, then G is called a **modular graph**.

- A **median graph** G is graph with the property that, $|I(u, v, w)| = 1$ for all triples of vertices u, v, w in G. An example is given by Fig:2

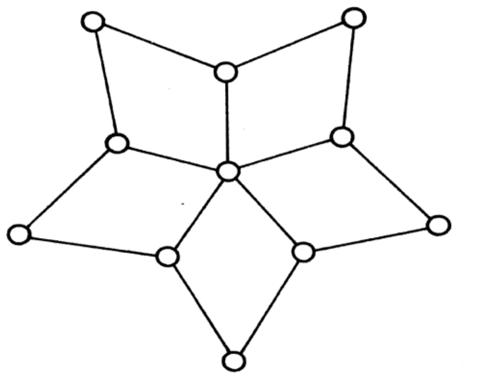

Fig: 2

- A ***gate*** for *u* in *W* is a vertex *x* in *W* such that $x\ \varepsilon\ I(u, w)$, for every vertex *w* in *W*. A set *W* is ***gated*** if every vertex has a unique gate in *W*.

- A ***profile*** on *G* is a sequence $\pi = x_{1\,,}\ x_{1\,,} x_{2\,,} \ldots .\, x_k$ of vertices of *G*. The ***length*** of the profile is $|\pi| = k$. We call π an *odd* profile when *k* is odd, and *even* when *k* is even,

- The **distance** of a vertex *v* to the profile π is $D(v, \pi) = \sum_{i=1}^{k} d(v,)$

- A ***median* or *median vertex*** of π is a vertex *x* minimizing the distance is $D(v, \pi)$ to the profile, and the *median set M* (π) of π is the set of all medians of π.

- **Convex Expansion:**

  Let $G' = (V', E')$ be properly covered by the convex subgraphs $G_1' = (V_1', E_1')$ and $G_2' = (V_2', E_2')$ and set $G_0' = (G_1' \cap G_2')$. For $i = 1{,}2$, let G_i be an isomorphic copy of G_i', and let λ_i be an isomorphism from G_i' onto G_i. We set $G_{0i} = \lambda_i[\,G_0]$ and $\lambda_i(u') = u_i$, for u' in G_0'. The expansion of G' with respect to the proper cover G_1', G_2' is the graph G obtained from the disjoint union of G_1 and G_2 by inserting an edge between u_1 in G_{01} and u_2 in G_{02}, for each $u'\ in\ G_0'$. Denote the set of edges between G_{01} and G_{02} by F_{12}. This is illustrated in Fig:3.

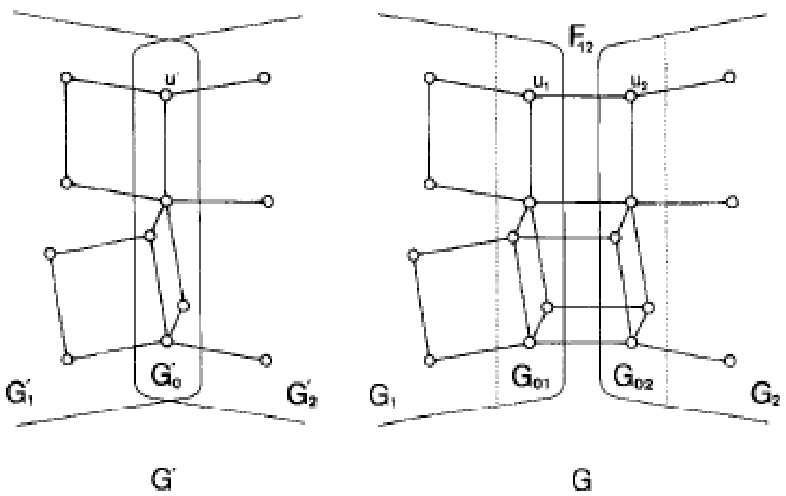

Fig:3

III. CHARACTERIZATIONS OF MEDIAN GRAPH

We have the following characterizations of median graph as proved in [7].

Theorem 1: A graph G is a median graph if and only if $|M(u,v,w)|$ = 1, for any triple of vertices u,v,w of G.

Theorem 2: A graph G is a median graph if and only if G can be obtained by successive expansions from the one vertex graph K_1 .

Theorem 3: If G is a median graph with split G_1 , G_2 then G_1 as well as G_2 , contains an extremal subgraph.

Theorem 4: A subgraph G_1 of a median graph G is extremal if and only if there is a split G_1 , G_2 and no color occurs only in G_1 .

Theorem 5: Let G be a median graph. Then G contains n pairwise crossing splits if and only if G contains an n-cube as an induced subgraph.

Theorem 6: A median graph G has no crossing splits if and only if G is a tree**.**

Theorem 7: A median graph is uniquely cutset colourable, up to the labelling of the colors.

Theorem 8: Let G be a graph that is not a hypercube. Then G is a median graph if and only if it can be obtained from two smaller median graphs by convex amalgamation.

Theorem 9: A graph G is a median graph if and only if it can be obtained from a set of hypercubes by convex amalgamations.

Theorem 10: A graph G is a median graph if and only if it can be isometrically embedded in a hypercube Q such that the median in Q of any three vertices in G lies also in G.

Theorem 11: Each automorphism of a median graph fixes some sub-cube, i.e. some regular median subgraph.

IV. APPLICATIONS

There are now numerous and varied uses for median graphics. Unsurprisingly, these are frequently generalizations of hypercube or tree applications that already exist. We quickly go over the most significant ones We were only able to provide a general notion of the complexity of median graph structure within the confines of this chapter. Only a few subjects, abstractions, and applications could be covered. Additionally, the bibliography is always only a limited sample of the body of material that already exists. Regarding median graphs, median structures, their generalizations, and applications, there is still considerable work to be done. They are used in location theory to find optimal locations. By definition, median graphs are interesting from the viewpoint of finding optimal locations. A consensus function is a model to describe a rational way to obtain consensus among a group of agents or clients. Median graphs are used to find consensus functions. In recent years a rather unexpected occurrence of median graphs in nature was discovered, e.g., in chemical substances. Median graphs are used to find the missing links in the history of evolution. They are also used in literary history to complete the missing links in transcription history. They also find significant uses in economics and voting theory,

V. CONCLUSION

Although median graphs are a very nice common generalization of trees and hypercubes, they have a very special structure. So it seems that the class of median graphs is quite esoteric and resides just somewhere in a remote corner of the universe of all graphs. A rich literature can be found on each of these median structures and their respective contexts. We were only able to provide a general notion of the complexity of median graph structure within the confines of this chapter. Only a few subjects, abstractions, and applications could be covered. Additionally, the bibliography is always only a limited sample of the body of material that already exists. Regarding median graphs, median structures, their generalisations, and applications, there is still considerable work to be done. This is a humble effort to bring to light the rich theory of median structures.

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