International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 11 Issue 1 ǁ January 2023 ǁ PP. 13-19

Further Results on Strongly Perfect Graphs

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ABSTRACT
In this paper, we prove – strongly perfect graphs are the line graph L(G) of (0,1), cartesian product of graphs, bipartite graphs, Cartesian product of non-trivial graphs and tensor product graph.
Key Words: *Induced subgraph, product graphs, cartesian product, Tensor product.*

Date of Submission: 22-12-2022 Date of acceptance: 03-01-2023

I. INTRODUCTION:

The strongly perfect graphs were first introduced by Claude Berge at a Monday Seminars, M.S.H., Paris, 1978. A graph is strongly perfect if each of its induced subgraphs H contains an independent set which meets all the cliques (maximal complete subgraphs) in H. That is a graph is strongly perfect if each of its induced subgraphs contain a good independent set. The strongly perfect graphs form an interesting class of perfect graphs because of the following:

1) Complement of a strongly perfect graphs not necessarily strongly perfect unlike the complement of a perfect graph.
2) In view of Ravindra's conjecture (33) that every p-critical graph is sp-critical, the strongly perfect graphs are closely related to solution of the famous unsettled Berge's Strong Perfect Graph Conjecture.
3) A strongly perfect graph serves as one of the best mathematical models for a situation where one would like to choose an optimal set of leaders from a given set of people (Ravindra (52)).

Berge and Duchet (6), Ravindra (31, 32), Chvatal (17) and Hoang (29) have obtained several interesting results in the area of strongly perfect graphs. We list all of these in the form of Facts as below for the sake of ready reference and completion.

FACT 1 (6, 31). Every P_4 – free graph is strongly perfect.

FACT 2 (6). Every triangulated graph is strongly perfect.

FACT 3. Every comparability graph is strongly perfect.

FACT 4 (6). A perfect graph G = (V, E) is strongly perfect iff no two families $C = (C_1, C_2, \ldots, C_k)$ and $D = (D_1, D_2, \ldots, D_k)$ of maximal cliques (with possible repeated cliques) satisfy

$|C| = |D|$ and $|C(v)| > |D(v)|$ for all $v \in V$. (C(v) is the sub-family of the cliques of C which contain v, D(v) has the similar meaning).

FACT 5 (32). If every odd cycle of length at least five in a graph G has at least two chords, then G is strongly perfect.

FACT 6 (61). The line graph L(G) of a graph G is strongly perfect if and only if each of the following properties is true.

i) Every block of G is either bipartite or K_{4-e} or $K_p (3 \le p \le 4)$.
ii) If C_r and C_s are two even cycles such that $V(C_r) \cap V(C_s) \neq \theta$, then $|V(C_r) \cap V(C_s)|$ is even.
iii) If C_i and C_j are two disjoint even cycles in G then all paths in G connecting C_i and C_j are of odd length.

FACT 7 (31). A line graph L (G) of G is stronglyperfect iff it does not contain C_{2n+1} $(n \ge 2)$ or any of the graphs in Fig. 2, 3 as an induced subgraph.

FACT 8 (31). For a total graph T(G) of G the following properties are equivalent.

i) T (G) is strongly perfect.
ii) T(G) is perfect
iii) Every block of G is either K_2 or K_3.

FACT 9 (34). Every strongly perfect B-graph contains a maximum and minimum good stable set.

Though Fact 4 gives a necessary and sufficient condition for a graph to be strongly perfect, there is no characterization of strongly perfect graphs in terms of forbidden subgraphs (that is, a complete set of sp-critical graphs is not known). However, some of the forbidden subgraphs are identified by Ravindra. Berge (Fig.1) Chvatal (Fig.2. personal communication to Ravindra), Maffray (Fig. 3, personal communication to Ravindra).

Further, with respect to $K_{1,3}$ – free graphs, Ravindra (33) has Conjectured that C_{2n+1} (n ≥ 2), $\overline{C}_n$ (n ≥ 5) and the graphs of the Fig.3 are the only sp-critical graphs.

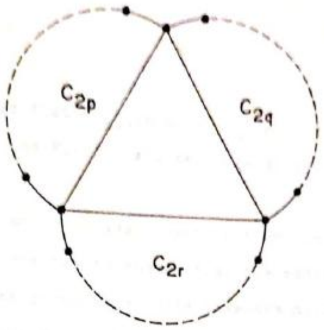

Fig. 1

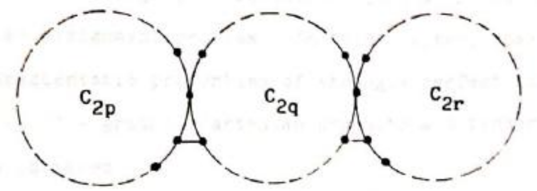

Fig. 2

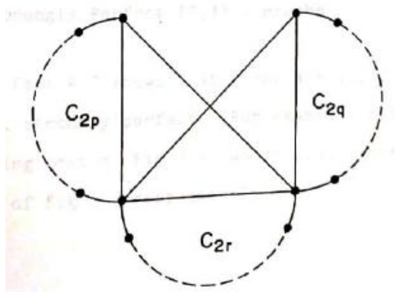

Fig. 3

To solve an assignment problem, in operations research, one has to ensure that the maximum number of independent zeros in an effectiveness matrix to be equal to the minimum number of lines covering all the zeros. Since (0, 1) – graphs are perfect, in view of the above fact, (0, 1) – graphs have direct link with the solution of an assignment problem.

2. Strongly Perfect (0,1) – graphs

Fact 7 shows that there are (0, 1) – graphs which are not strongly perfect. For example, consider the following graphs (4) as special cases of G_1, G_2 and G_3 of Fig. 3 respectively.

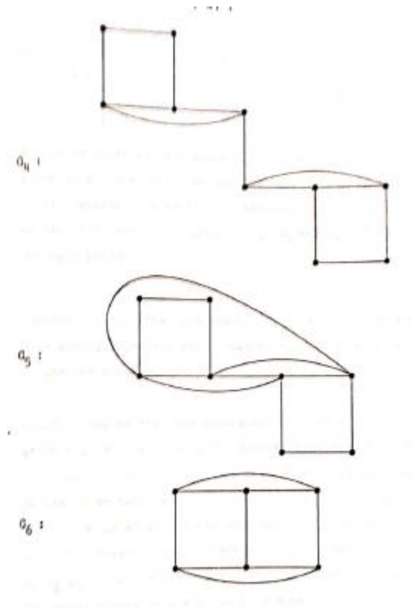

Fig 4

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

It can be verified that graph G_4 is the (o, 1) – graph of A. Similarly, we can see that G_5 and G_6 are (o, 1) – graphs of some (0, 1) – matrices. Therefore, by Fact 7, the (0, 1) – graphs G_4, G_5 and G_6 are not strongly perfect.

THEOREM 2.1. : The line graph L (G) of a (0,1) – graph G is strongly perfect iff it has no C_{2n+1} $(n \geq 2)$ an induced subgraph.

Proof: Suppose that the line graph L (G) of a (o,1) – graph G is strongly perfect. Then by virtue of Fact 7, L(G) does not contain C_{2n+1} $(n \geq 2)$ as an induced subgraph. On the other hand, suppose that L(G) does not contain C_{2n+1} $(n \geq 2)$ as an induced subgraph. If L (G) is not strongly perfect, then by Fact 7, L(G) contains G_1, G_2 or G_3 as an induced subgraph. That is G contains the following graphs (Fig.5) as subgraphs.

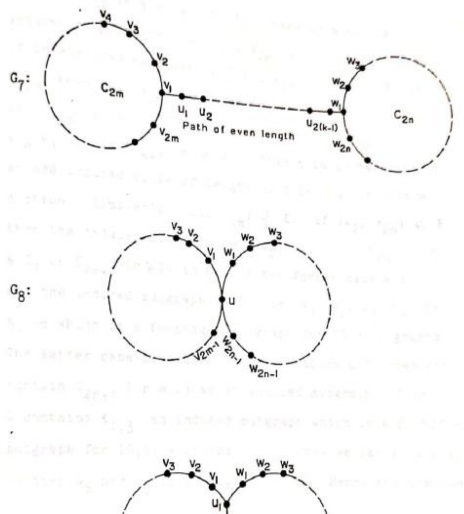

Fig. 5

If G_7 is a subgraph of G, then we show that the induced subgraph on V_1, V_2, V_{2m}, u_1, is $K_{1,3}$. For, if the induced subgraphon V_1, V_2, V_{2m}, u_1, is not $K_{1,3}$. Then $(u_1, V_{2m}) \in\geq$ E.or$(u_1, V_2) \in$ E or $(v_2, v_{2m}) \in$ E. If $(u_1, v_2) \in$ E then $u_1, v_2, v_3, \ldots V_{2m}, v_1 u_1$ is C_{2m+1} ($m \geq 2$). Then this corresponds to an odd induced cycle of length ≥ 5 in L(G), a contradiction. Similarly, $(u_1, v_{2m}) \notin$ E. If $(v_2, v_{2m}) \in$ E then the induced subgraph on $v_2, v_3, \ldots \ldots, v_{2m}$ is a C_3 or C_{2r+1} ($r \geq 2$) in G. In the former case m = 2 and the induced subgraph of G7 on V_1, V_2, V_3, V_4 is K_4-E which is a forbidden subgraphfor 0.1 graphs. The later case also does not arise since L(G) does not contain C_{2r+1} ($r \geq 2$) as an induced subgraph. Thus, G contains $K_{1,3}$ as induced subgraph which is a forbidden subgraph for (0, 1) – graphs. Similarly we can show that neither G_8 nor G_9 is a subgraph of G. Hence the theorem.

As an immediate consequence of the above theorem, we have the following.

COROLLARY 2.1. : A perfect line graph of a (0, 1) – graph is strongly perfect.

3. Strongly Perfect Product Graphs.

Ravindra and Parthasarathy (36) studied at length the perfectness of Normal products, Cartesian products, Tensor products and Lexicographic products (compositions) and obtained necessary and sufficient conditions for the three latter products to be perfect.

The following Facts due to Ravindra and Parthasarathy (36) characterize perfect Cartesian and tensor products.

FACT3.1. G_1 X G_2 is perfect iff it has no induced odd cycle of length at least five.

FACT 3.2. G_1 X G_2 is perfect iff one of the following holds.

i) G_1 and G_2 are bipartite.

ii) G_1 or G_2 is Meyniel and Z-free and the other is K_2. (Here Z is the graph of Fig.6).

iii) G_1 or G_2 is $K_{t_1,t_2,\ldots t_r}$ Kt1, ($r \geq 3$ and $t \geq 2$ for some i) and the other is a tree.

iv) Every block of G_1 or G_2 is complete and the other is a complete graph.

FACT 3.3. $G_1 \wedge G_2$ is perfect iff either

i) G_1 or G_2 is bipartite, or

ii) both G_1 and G_2 are Y-free Berge (where Y is the graph of Fig.2.2)

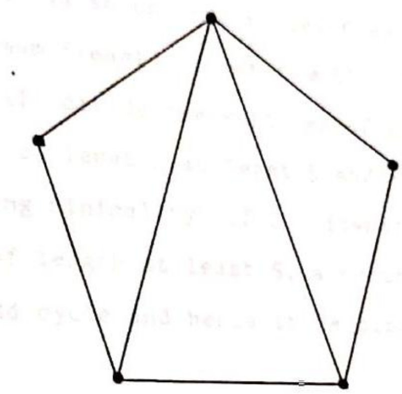

Fig. 6

Here we characterise strongly perfect Cartesian and tensor product graphs.

Some of the observations made earlier or elsewhere are useful in proving the following theorems and we list them as lemmas.

LEMMA 3.1. If G is a triangle fee graph and if it does not contain an odd induced cycle of length at least five, then G is bipartite.

Proof: If G has an odd cycle, let C be an odd cycle of G with minimum (least) length (≥ 5). If C has a chord e, then e will divide the vertices of C into an odd and even cycles of lengths at least 5 and 4 respectively, contradicting minimality of C. Therefore, C is an induced odd cycle of length at least 5, a contradiction. Thus G has no odd cycle and hence it is bipartite.

LEMMA 3.2. $\overline{C}_n$ $n \geq 5$ is not strongly perfect.

Proof: Let $V_1 V_2 \ldots V_n$ be the cycle C_n. Let $e_i = V_i V_{i+1}$ (subscripts taken addition modulo n) be an edge of C. e cannot meet the maximal independent set containing $V_{i-1} V_{i+2}$. This implies that no edge of C_n meets all the maximal independent sets in it. That is, $\overline{C}_n$ does not contain an independent set which meets all the cliques in it. Thus G is not strongly perfect.

THEOREM 3.1. Let G be a Cartesian product of graphs other than K_1. Then G is strongly perfect if and only if it is bipartite.

Proof : Since a bipartite graph is obviously a strongly perfect graph, it is enough to prove that strongly perfect Cartesian product of non-trivial graph is bipartite.

Let G = G_1 X G_2 where G is strongly perfect and G_1 and $G_2$are some graphs other than K_1. We now claim that G_1 and G_2 are bipartite. If G_1 and G_2 are not bipartite, let without loss of generality G_1 not be a bipartite graph. Then by Lemma 3.1 G_1 contains a K_3 as G_1 is strongly perfect. Since G_2 is non trivial, it contains a K_2. But then G contains K_3 X K_2 = $\overline{C}_6$ (Fig.7) as an induced subgraph, a contradiction in view of Lemma 3.2, establishing G_1 and G_2 are bipartite.

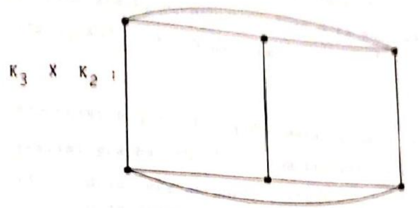

Fig. 7

Let $V_1 = V_{11} \cup V_{12}$ and $V_2 = V_{21} \cup V_{22}$ be, respectively bipartitions of G_1 and G_2. By definition of Cartesian product of graphs, it is not difficult to verify that G is a bipartite graph with bipartition
$V = (V_{11} \text{ x } V_{21} \cup V_{12} \text{ X } V_{22}) \cup (V_{11} \text{ x } V_{22} \cup V_{12} \text{ X } V_{21})$.

Since the Cartesian product G =G_1 X G_2 is bipartite iff G_1 and G_2 are bipartite, the above theorem implies immediately.

COROLLARY 3.1: Let G be a Cartesian product of non-trivial graphs G_1 and G_2. Then G is strongly perfect iff G_1 and G_2 are bipartite.

COROLLARY 3.2: For a Cartesian product G of non trivial graphs, the following properties are equivalent.

i) G is very strongly perfect.
ii) G is strongly perfect.
iii) G is bipartite.
iv) G is perfectly orderable.

Proof : i) $\Rightarrow$ ii) (by definition)
ii) $\Rightarrow$ iii) (by Theorem 3.1)

To prove iii) $\Rightarrow$iv), assume that G is a bipartite graph with bipartition V = U $\cup$ W. Define a linear order < on V by x < y iff x $\in \cup$ and y $\in$ W. If abed is a P_4 in G with a $\in$ U, then by the definition of <.a< b and c < d (since G is bipartite). Thus V admits a linear order < such that no induced P_4 has a < b, d < c and hence G is perfectly orderable. Iv) $\Rightarrow$ii) by the Fact 9. Since iv) $\Rightarrow$ ii), ii) $\Rightarrow$ iii) and iii) $\Rightarrow$ i) (as every bipartite graph is very strongly perfect), it immediately follows that iv) $\Rightarrow$ i).

THEOREM 3.2. A tensor product graph is strongly perfect if and only if it is bipartite.

Proof: Suppose G = $G_1 \wedge G_2$ is strongly perfect graph. If G_1 or G_2 is K_1, then all the vetices of G are isolated vertices, in which case G is obviously bipartite. Since $K_3 \wedge K_3$ contains $\overline{C}_6$ (1 2 3 4 5 6) as an induced subgraph (Fig. 4, 8), G_1 or G_2 is triangle free. Let without loss of generality G_1 be triangle free.

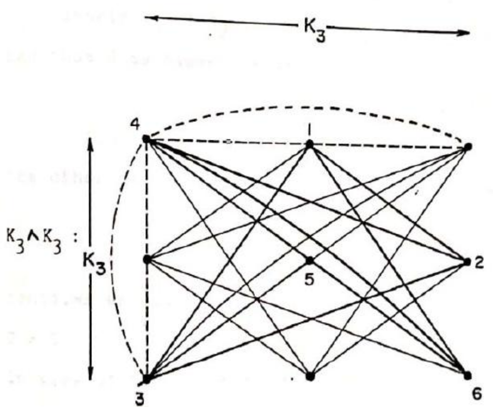

Fig. 8

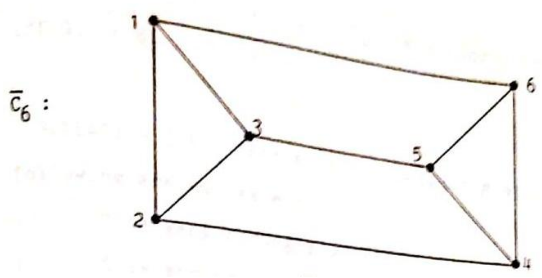

Fig. 9

G_1, being a strongly perfect graph, does not contain an odd induced cycle of length at least 5. Hence by Lemma 3.2, G_1 is bipartite. Let $V_1 = V_{11} \cup V_{12}$ be a bipartition of G_1. By definition of tensor product of G_1 and G_2 clearly V_{11} x V_2 and V_{12} x V_2 are independent sets and thus G is bipartite graph.

Since every bipartite graph is strongly perfect, the other part of the theorem follows immediately.

Since the tensor product of any two odd cycles contains an odd cycle, it follows immediately that $G = G_1 \wedge G_2$ is bipartite iff G_1 or G_2 is bipartite. In view of this, the above theorem can be restated as follows:

COROLLARY 3.3. $G = G_1 \wedge G_2$ is strongly perfect iff G_1 or G_2 is bipartite.

COROLLARY 3.4. For a tensor product graph, G the following are equivalent.

i) G is very strongly perfect.
ii) G is strongly perfect.
iii) G is bipartite graph.
iv) G is perfectly orderable.

Proof of this Corollary is similar to that of Corollary 3.2.

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