

An interesting subclass constructed using a new parameter of an already defined class of regular functions and its coefficient inequality

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Abstract –In this paper, we solve an inequality proved by Fekete and Szegő[5] in 1933 by using the analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for a new class.

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I. Introduction

We are dealing with geometric function theory, it is that branch of complex analysis which deals with the analytic functions geometrically. The pillar of this theory is Riemann Mapping Theorem which was proved in 19th century. It initiated its roots in the work of great mathematician Koebe [19] in 1907, who stated that "An analytic function which is univalent has properties of conformal mapping i.e. angle preserving property". From this theorem, Bieberbach conjecture was proved. This was given by L. Bieberbach[2] in 1916 but proved finally by Louis De Branges [3] in 1985 and while tackling with this conjecture, an equality arises, which is called FeketeSzegő Inequality given by Fekete and Szegő [5].

The inequality which is for the function $f(z) \in A$ and based on Bieberbach conjecture, is named as FeketeSzegő Inequality, which states that if $f(z)$ is a function of type

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which is univalent in E , then

$$|a_3 - \alpha a_2^2| \leq \begin{cases} 3 - 4\alpha & \text{if } \alpha \leq 0 \\ 1 + 2 \exp\left(\frac{-2\alpha}{1-\alpha}\right) & \text{if } 0 \leq \alpha \leq 1 \\ 4\alpha - 3 & \text{if } \alpha \geq 1 \end{cases}$$

This is an inequality which is related to univalent analytic functions [18] - [40] and gives the necessary condition to map the unit disk of a complex plane injectively to the complex plane. It gives the relation between second and third coefficient of univalent analytic function.

In order to prove our result, let us explain some classes and some basic results related to our work:-

A consists all those functions f which are analytic in open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and are of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with normalization conditions $f(0) = 0, f'(0) = 1$.

S be the family of functions f which are univalent in the open disk $\{z \in \mathbb{C} : |z| < 1\}$ with conditions $f(0) = 0, f'(0) = 1$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

$S^*(\phi)$ be the class of functions in $f \in S$, for which $\frac{z f'(z)}{f(z)} < \phi(z)$, given by Ma and Minda [10].

$t(z)$ be a family of analytic functions in the open unit disk E , having functions of the form $t(z) = \sum_{n=1}^{\infty} c_n z^n$, it is a class of bounded analytic function denoted by U , if the conditions

$t(0) = 0$ and $|t(z)| < 1$ hold. The necessary conditions for any function to be bounded analytic function are $|c_1| \leq 1, |c_2| \leq 1 - |c_1|^2$; which were given by Miller et. al. [11],

Let $u(z)$ and $v(z)$ are two analytic functions in E . If there exists a Schwarzian function $F(z)$ (analytic in E) in such a way that $|F(z)| < 1, F(0) = 0$ and $u(z) = v(F(z)) ; z \in E$ then the function $u(z)$ is subordinate to $v(z)$ written as $u(z) < v(z)$ and this concept (called subordination) was given by Lindelof [9].

We introduce a new class $KS^*(f, f', f'')$ of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; defined as

$$\alpha \left(\frac{z \{ (f'(z))^2 + f(z)f''(z) \}}{f(z)f'(z)} \right) + (1-\alpha) \frac{\{z\{zf'(z)\}'\}}{\{zf'(z)\}'} = \frac{1+w(z)}{1-w(z)}; z \in E \dots (1.1)$$

II. Main Results:-

THEOREM1:- Let $f(z) \in KS^*(f, f', f'')$ and $\phi(z) = \left(\frac{1+z}{1-z}\right)^\delta$; $w(z)$ is a Schwarzian function ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta^2}{2} - \mu\delta^2; \mu \leq \frac{(\delta-1)}{2\delta}; \\ \frac{\delta}{2}; \frac{(\delta-1)}{2\delta} \leq \mu \leq \frac{(\delta+1)}{2\delta}; \\ \mu\delta^2 - \frac{\delta^2}{2}; \mu \geq \frac{(\delta+1)}{\delta}. \end{cases}$$

PROOF:- By definition of $KS^*(f(z), f(az))$, given by (1.1)

and using $w(z) = c_1z + c_2z^2 + c_3z^3 + \dots$, $f(z) = z + a_2z^2 + a_3z^3 + \dots$

$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \dots$, we get

we get

$$1 + 4a_2z + (6a_2^2 + 6a_3)z^2 + \dots = 1 + (2a_2 + 2\delta c_1)z + (2a_3 + 2a_2^2 + 4a_2\delta c_1 + 2\delta(c_2 + \delta c_1^2))z^2 + \dots$$

Comparing like coefficients, one can easily obtain

$$a_2 = \delta c_1 \text{ and } a_3 = \frac{1}{2}(\delta^2 c_1^2 + \delta c_2)$$

Using these values of a_2 and a_3 , one can construct

$$a_3 - \mu a_2^2 = \frac{\delta c_2}{2} + \left(\frac{\delta^2}{2} - \delta^2 \mu\right) c_1^2$$

After applying modulus on both sides, we get

$$|a_3 - \mu a_2^2| \leq \left(\frac{\delta}{2}\right) |c_2| + \delta^2 \left|\frac{1}{2} - \mu\right| |c_1|^2$$

Using $|c_2| \leq 1 - |c_1|^2$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{2} + \left\{ \delta^2 \left| \frac{1}{2} - \mu \right| - \frac{\delta}{2} \right\} |c_1|^2$$

Case 1:- If $\mu \leq \frac{1}{2}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{\delta}{2} + \left\{ \frac{\delta(\delta-1)}{2} - \mu \delta^2 \right\} |c_1|^2$$

Subcase - 1 (a):- When $\mu \leq \frac{(\delta-1)}{2\delta}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta^2}{2} - \mu\delta^2 \dots \dots \dots (1.2)$$

Subcase - 1 (b):- When $\mu \geq \frac{(\delta-1)}{2\delta}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{\delta}{2} \dots \dots \dots (1.3)$$

Case - 2:- If $\mu \geq \frac{1}{2}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{\delta}{2} + \left\{ \mu\delta^2 - \frac{(\delta^2+\delta)}{2} \right\} |c_1|^2$$

Subcase-2 (a):- When $\mu \geq \frac{(\delta+1)}{2\delta}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \mu\delta^2 - \frac{\delta^2}{2} \dots \dots \dots (1.4)$$

Subcase - 2 (b):- When $\mu \leq \frac{(\delta+1)}{2\delta}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{\delta}{2} \dots \dots \dots (1.5)$$

Combining (1.2), (1.3), (1.4) and (1.5), we get the required result.

Corollary 2:- $KS^*(f, f', f'') = KS_p^*$, as by substituting $\delta=1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2} - \mu; \mu \leq 0; \\ \frac{1}{2}; 0 \leq \mu \leq 1; \\ \mu - \frac{1}{2}; \mu \geq 1. \end{cases}$$

which is the required result given by Gurmeet Singh, Gagandeep Singh and M. S. Arora [18].

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