International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 10 Issue 7 ǁ July 2022 ǁ PP. 272-276

Generalized Holder's inequalities for extended Chaudhary-Zubair gamma function

Omprakash Atale
*[*1]Department of Mathematics,*
Khandesh College Education Society's
MooljiJaitha College
"An Autonomous College Affiliated to K.B.C.
North Maharashtra University, Jalgaon."
Corresponding Author Email:atale.om@outlook.com

Abstract
The aim of this paper is to derive some Holder's inequalities for the extended Chaudhary-Zubair gamma function, which is also known as the j-m Chaudhary-Zubair gamma function. Furthermore, some refinements of Holder's inequality are also derived.

Keywords:_Holder's inequalities, Gamma function, Extended gamma function._

Date of Submission: 01-07-2022 Date of acceptance: 11-07-2022

I. INTRODUCTION

The aim of this paper is to derive some inequalities of the extended Chaudhary-Zubair gamma function which is also known as the *j-m*Chaudhary-Zubair gamma function. To be specific, we are going to apply the generalizations of the Holder's inequality derived by Mohamed Akkouchi and Mohamed Amine Ighachanein their paper [1]. Furthermore, using their generalizations of the Holder's inequality, some new refinements of the Holder's inequality are also established. In [2], Chaudhary and Zubair gave the following extension of the gamma function:

$$\Gamma_{\omega}(x)=\int_0^\infty t^{x-1}e^{-t-\frac{\omega}{t}}.$$

The above extension of the gamma function has wide applications in mathematics in the area of statistical distributions. Since then, many generalizations of 1.1 have been derived in the literature. Consider for example the following *m*-analogue of the Chaudhary-Zubair gamma function [3]

$$\Gamma_{\omega,m}(x)=\int_0^\infty t^{x-1}e^{-\frac{t^m}{m}-\frac{\omega^m}{mt^m}}dt$$

and the *j-m* analogue:

$$\Gamma_{\omega,m,j}(x)=\int_0^\infty t^{x-1}e^{-\frac{t^m}{j}-\frac{\omega}{\frac{t^m}{j}}}dt$$

where $\omega \geq 0$ and $j,m>0$.

II. Main results

Let ξ and $\tilde{\xi}$ be two non-negative integrable functions over the range of $[a,b]$. Let $p,q>0$ such that $p^{-1}+q^{-1}=1$. Then, we define Holder's inequality as

$$\left(\int_a^b \xi^p(t)dt\right)^{\frac{1}{p}}\left(\int_a^b \tilde{\xi}^q(t)dt\right)^{\frac{1}{q}} \geq \int_a^b \xi(t)\tilde{\xi}(t)dt.$$

For the particular case of $p=q=2$, the above inequality is reduced to the Cauchy-Schwarz inequality. Akkouchi&Ighachane gave the following refinement of the above Holder's inequality:

Theorem 1: *Let ξ and $\tilde{\xi}$ be two non-negative integrable functions over the range of $\Omega:[a,b]$. Let $p,q>0$ such that $p^{-1}+q^{-1}=1$. Then for $n\geq 2$ we have*

$$\int_{\Omega}\left|\xi(x)\tilde{\xi}(x)\right|d\mu(x)\leq\left(\frac{1}{p^n}+\frac{1}{q^n}\right)\|\xi\|_p\|\tilde{\xi}\|_q+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^kq^{n-k}}\|\xi\|_p^{1-\frac{kp}{n}}\|\tilde{\xi}\|_q^{1-\frac{(n-k)q}{n}}\int_{\Omega}|\xi|^{\frac{kp}{n}}\left|\tilde{\xi}\right|^{\frac{(n-k)q}{n}}d\mu(x)$$

$$\leq\left(\int_{\Omega}|\xi|^p\,d\mu(x)\right)^{\frac{1}{p}}\left(\int_{\Omega}\left|\tilde{\xi}\right|^q d\mu(x)\right)^{\frac{1}{q}}$$

where $\binom{n}{k}=\dfrac{n!}{(n-k!)k!}$ *is the usual binomial coefficient for all* $k\in\{0,1,2,...,n\}$.

Furthermore, they also gave a equivalence of Holder's and Cauchy-Schwarz inequalities as follows.

Theorem 2: *Let ξ and $\tilde{\xi}$ be two non-negative integrable functions over the range of $\Omega:[a,b]$. Let $p,q>0$ such that $p^{-1}+q^{-1}=1$. Then we have*

$$\int_{\Omega}\left|\xi(x)\tilde{\xi}(x)\right|d\mu(x)\leq\left(1-\frac{2}{pq}\right)\|\xi\|_p\|\tilde{\xi}\|_q+\frac{2}{pq}\|\xi\|_p^{1-\frac{p}{2}}\|\tilde{\xi}\|_q^{1-\frac{q}{2}}\int_{\Omega}|\xi|^{\frac{p}{2}}\left|\tilde{\xi}\right|^{\frac{q}{2}}d\mu(x)$$

$$\leq\left(\int_{\Omega}|\xi|^p\,d\mu(x)\right)^{\frac{1}{p}}\left(\int_{\Omega}\left|\tilde{\xi}\right|^q d\mu(x)\right)^{\frac{1}{q}}.$$

Now, we apply Theorem 1to *m*-analogue and *j*-*m*-analogue of the Chaudhary-Zubair gamma function.

Theorem 3: *Let $\omega\geq 0$. Let $a,b>0$ with $a+b=1$ and $u,v>0$. Then for $n\geq 2$ we have*

$$\Gamma_{\omega,m}(au+bv)$$

$$\leq\left(a^n+b^n\right)\Gamma_{\omega,m}(u)^a\,\Gamma_{\omega,m}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^kb^{n-k}\Gamma_{\omega,m}(u)^{a-\frac{k}{n}}\,\Gamma_{\omega,m}(v)^{b-\frac{n-k}{n}}\,\Gamma_{\omega,m}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)$$

$$\leq\Gamma_{\omega,m}(u)^a\,\Gamma_{\omega,m}(v)^b.$$

Proof: To apply Theorem 1 to the *k*-analogue of Chaudhary-Zubair gamma function, we first set and let the measure μ given by $d\mu(t)=e^{-\frac{t^m}{m}-\frac{\omega^m}{mt^m}}dt$. Now, let $\xi(x)=x^{a(u-1)}$ and $\tilde{\xi}(x)=x^{a(v-1)}$ and $p=\frac{1}{a},q=\frac{1}{b}$ which implies that $a+b=1$. Thus, now we get

$$\|\xi\|_p=\left(\int_0^{\infty}x^{(u-1)}d\mu(x)\right)^a=\Gamma_{\omega,m}(u)^a,$$

$$\|\tilde{\xi}\|_p=\left(\int_0^{\infty}x^{(v-1)}d\mu(x)\right)^b=\Gamma_{\omega,m}(v)^b$$

and

$$\int_0^\infty x^{a(u-1)}x^{b(v-1)}d\mu(x)=\Gamma_{\omega,m}(au+bv).$$

Using theorem 1, we have

$$\Gamma_{\omega,m}(au+bv)$$

$$\le\left(a^n+b^n\right)\Gamma_{\omega,m}(u)^a\,\Gamma_{\omega,m}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^kb^{n-k}\Gamma_{\omega,m}(u)^{a-\frac{k}{n}}\,\Gamma_{\omega,m}(v)^{b-\frac{n-k}{n}}\int_0^\infty x^{\frac{k}{n}(u-1)}x^{\frac{(n-k)}{n}(v-1)}d\mu(x)$$

$$=\left(a^n+b^n\right)\Gamma_{\omega,m}(u)^a\,\Gamma_{\omega,m}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^kb^{n-k}\Gamma_{\omega,m}(u)^{a-\frac{k}{n}}\,\Gamma_{\omega,m}(v)^{b-\frac{n-k}{n}}\,\Gamma_{\omega,m}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right).$$

Since $\Gamma_{\omega,m}(x)$ is logarithmically convex, we get

$$\Gamma_{\omega,m}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)\le\Gamma_{\omega,m}(u)^{\frac{k}{n}}\,\Gamma_{\omega,m}(v)^{\frac{(n-k)}{n}}$$

for all $1\le k\le n-1$. We have

$$\Gamma_{\omega,m}(au+bv)$$

$$\le\left(a^n+b^n\right)\Gamma_{\omega,m}(u)^a\,\Gamma_{\omega,m}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^kb^{n-k}\Gamma_{\omega,m}(u)^{a-\frac{k}{n}}\,\Gamma_{\omega,m}(v)^{b-\frac{n-k}{n}}\,\Gamma_{\omega,m}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)$$

$$\le\Gamma_{\omega,m}(u)^a\,\Gamma_{\omega,m}(v)^b.$$

This completes our proof.

Theorem 4: *Let* $\omega\ge 0$. *Let* $a,b>0$ *with* $a+b=1$ *and* $u,v>0$. *Then for* $n\ge 2$ *we have*

$$\Gamma_{\omega,m,j}(au+bv)$$

$$\le\left(a^n+b^n\right)\Gamma_{\omega,m,j}(u)^a\,\Gamma_{\omega,m,j}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^kb^{n-k}\Gamma_{\omega,m,j}(u)^{a-\frac{k}{n}}\,\Gamma_{\omega,m,j}(v)^{b-\frac{n-k}{n}}\,\Gamma_{\omega,m,j}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)$$

$$\le\Gamma_{\omega,m,j}(u)^a\,\Gamma_{\omega,m,j}(v)^b.$$

Proof: To apply Theorem 1 to the *j*-*m*-analogue of Chaudhary-Zubair gamma function, we first set and let the measure μ given by $d\mu(t)=e^{-\frac{t^m}{j}-\frac{\omega}{\frac{t^m}{j}}}dt$. Now, let $\xi(x)=x^{a(u-1)}$ and $\tilde{\xi}(x)=x^{a(v-1)}$ and $p=\frac{1}{a},q=\frac{1}{b}$ which implies that $a+b=1$. Thus, now we get

$$\|\xi\|_p=\left(\int_0^\infty x^{(u-1)}d\mu(x)\right)^a=\Gamma_{\omega,m,j}(u)^a,$$

$$\|\tilde{\xi}\|_p=\left(\int_0^\infty x^{(v-1)}d\mu(x)\right)^b=\Gamma_{\omega,m,j}(v)^b$$

and

$$\int_0^\infty x^{a(u-1)} x^{b(v-1)} d\mu(x) = \Gamma_{\omega,m,j}(au+bv)$$

Using theorem 1, we have

$$\Gamma_{\omega,m,j}(au+bv)$$
$$\leq \left(a^n+b^n\right)\Gamma_{\omega,m,j}(u)^a\,\Gamma_{\omega,m,j}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^k b^{n-k}\Gamma_{\omega,m,j}(u)^{a-\frac{k}{n}}\Gamma_{\omega,m,j}(v)^{b-\frac{n-k}{n}}\int_0^\infty x^{\frac{k}{n}(u-1)}x^{\frac{(n-k)}{n}(v-1)}d\mu(x)$$
$$=\left(a^n+b^n\right)\Gamma_{\omega,m,j}(u)^a\,\Gamma_{\omega,m,j}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^k b^{n-k}\Gamma_{\omega,m,j}(u)^{a-\frac{k}{n}}\Gamma_{\omega,m,j}(v)^{b-\frac{n-k}{n}}\Gamma_{\omega,m,j}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)$$

.

Since $\Gamma_{\omega,m,j}(x)$ is logarithmically convex, we get

$$\Gamma_{\omega,m,j}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)\leq\Gamma_{\omega,m,j}(u)^{\frac{k}{n}}\,\Gamma_{\omega,m,j}(v)^{\frac{(n-k)}{n}}$$

for all $1\leq k\leq n-1$. Thus, we get

$$\Gamma_{\omega,m,j}(au+bv)$$
$$\leq \left(a^n+b^n\right)\Gamma_{\omega,m,j}(u)^a\,\Gamma_{\omega,m,j}(v)^b+\sum_{k=1}^{n-1}\binom{n}{k}a^k b^{n-k}\Gamma_{\omega,m,j}(u)^{a-\frac{k}{n}}\Gamma_{\omega,m,j}(v)^{b-\frac{n-k}{n}}\Gamma_{\omega,m,j}\left(\frac{k}{n}u+\frac{(n-k)}{n}v\right)$$
$$\leq\Gamma_{\omega,m,j}(u)^a\,\Gamma_{\omega,m,j}(v)^b.$$

This completes our proof.

Now, we present the following two refinements of the Holder's inequality derived from Theorem 1 and 2.

Theorem 5: *Let ψ and φ be two non-negative integrable functions over the range of $\Omega:[a,b]$. Let $p,q>0$ such that $p^{-1}+q^{-1}=1$. Let $m,n>0$ such that $\frac{m}{p}+\frac{n}{q}$ is an integer. Then we have*

$$\int_\Omega\left|\psi\varphi^{\frac{m}{p}+\frac{n}{q}}\right|d\mu(x)\leq\left(1-\frac{2}{pq}\right)\left\|\psi^{\frac{1}{p}}\varphi^{\frac{m}{p}}\right\|_p\left\|\psi^{\frac{1}{q}}\varphi^{\frac{n}{q}}\right\|_q+\frac{2}{pq}\left\|\psi^{\frac{1}{p}}\varphi^{\frac{m}{p}}\right\|_p^{1-\frac{p}{2}}\left\|\psi^{\frac{1}{q}}\varphi^{\frac{n}{q}}\right\|_q^{1-\frac{q}{2}}\int_\Omega\left|\psi\varphi^m\right|^{\frac{1}{2}}\left|\psi\varphi^n\right|^{\frac{1}{2}}d\mu(x)$$
$$\leq\left(\int_\Omega\left|\psi\varphi^m\right|d\mu(x)\right)^{\frac{1}{p}}\left(\int_\Omega\left|\psi\varphi^n\right|d\mu(x)\right)^{\frac{1}{q}}.$$

Proof: Replace ξ and $\tilde{\xi}$ in Theorem 2 with $\psi^{\frac{1}{p}}\varphi^{\frac{m}{p}}$ and $\psi^{\frac{1}{q}}\varphi^{\frac{n}{q}}$ respectively and the desired result readily follows.

Theorem 6: *Let ψ and φ be two non-negative integrable functions over the range of $\Omega:[a,b]$. Let $p,q>0$ such that $p^{-1}+q^{-1}=1$. Let $m,n>0$ such that $\frac{m}{p}+\frac{n}{q}$ is an integer. Then for $n\geq 2$ we have*

$$\int_{\Omega}\left|\psi\varphi^{\frac{m}{p}+\frac{n}{q}}\right|d\mu(x)\leq\left(\frac{1}{p^{n}}+\frac{1}{q^{n}}\right)\left\|\psi^{\frac{1}{p}}\varphi^{\frac{m}{p}}\right\|_{p}\left\|\psi^{\frac{1}{q}}\varphi^{\frac{n}{q}}\right\|_{q}+\sum_{k=1}^{j-1}\binom{n}{k}\frac{1}{p^{k}q^{n-k}}\left\|\psi^{\frac{1}{p}}\varphi^{\frac{m}{p}}\right\|_{p}^{1-\frac{kp}{j}}\left\|\psi^{\frac{1}{q}}\varphi^{\frac{n}{q}}\right\|_{q}^{1-\frac{(n-k)q}{j}}$$

$$\int_{\Omega}\left|\psi\varphi^{m}\right|^{\frac{k}{j}}\left|\psi\varphi^{n}\right|^{\frac{(j-k)}{j}}d\mu(x)\leq\left(\int_{\Omega}\left|\psi\varphi^{m}\right|d\mu(x)\right)^{\frac{1}{p}}\left(\int_{\Omega}\left|\psi\varphi^{n}\right|d\mu(x)\right)^{\frac{1}{q}}.$$

Proof: Replace ξ and $\tilde{\xi}$ in Theorem 1 with $\psi^{\frac{1}{p}}\varphi^{\frac{m}{p}}$ and $\psi^{\frac{1}{q}}\varphi^{\frac{n}{q}}$ respectively and the desired result readily follows. .

III. CONCLUSION

In this paper, we have derived some inequalities for the extended Chaudhary-Zubair gamma functions.Furthermore, using the results of Mohamed Akkouchi and Mohamed Amine Ighachane, we have derived some refinements of the Holder's inequality in the form of Theorem 5 and 6. The substitutions that we have used in deriving Theorem 5 and 6 were previously useful in deriving the Turan-type inequalities for polygama function and other special functions [5]. Thus, we believe that Theorem 5 and 6 can also be used in deriving Turan-type inequalities for the same special functions.

REFERENCES

[1]. Akkouchi, Mohamed &Ighachane, Mohamed. (2020). Some refinements to Hölder's inequality and applications. Proyecciones(Antofagasta). 39. 153-166.

[2]. Chaudhry MA, Zubair SM, Generalized incomplete gamma functions with applications, Journalof Computational and Applied Mathematics, 55(1994), 99-124 [3].

[3]. Mubeen, Shahid and Purohit, Sunil and Arshad, Muhammad and Rahman, Gauhar. (2016).EXTENSION OF k-GAMMA, k-BETA FUNCTIONS AND k-BETA DISTRIBUTION. Journalof Mathematical Analysis. 7. 118-131.

[4]. E.F. Beckenbach and R. Bellman, Inequalities, Springer Verlag, Berlin, 1961.

Mortici, Cristinel. (2010). TURN-TYPE INEQUALITIES FOR THE GAMMA ANDPOLYGAMMA FUNCTIONS. Acta Universitatis Apulensis. Mathematics - Informatics. 23.