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Mass Matrix of 2-D Pipe Elbow Element

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Abstract
The dynamic analysis of the piping system is significant, as most of the piping loads are time dependent and the equivalent static analysis method does not provide information for predicting fatigue failures, especially when the loads are acting simultaneously. The primary step in the dynamic analysis is to determine the Mass matrix of the piping system. The finite element model of the piping system consists of pipe spools, pipe elbows and other elements classified as rigid elements. The pipe spools can be considered as beam elements and mass matrix of beam elements are widely available in the literature, but pipe elbows are modeled using equivalent lengths of piping. In this paper, the mass matrix of the pipe elbow is determined from first principles of virtual work and the mass matrix coefficients are tabulated for standard pipe elbow sizes.

Keywords: Mass Matrix, Pipe Elbow, Shape Factor, Finite Element Analysis

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I. INTRODUCTION

In the dynamic analysis of the piping system, the combined mass matrix is computed by assembling the mass matrices of individual piping elements. Mass matrix of the pipe elbow, which is a part of the piping system, is derived using the virtual work method in this paper. The method of determination of mass matrix for 2D pipe elbow is similar to that of the method used for determining mass matrix of beam elements. But pipe elbow mass matrix evaluation is more complex than the straight beam elements due to its geometry. In the below sections, the method used for derivation of mass matrix of pipe elbow is discussed in detail.

II. FORMULATION FOR MASS MATRIX

2.1 METHOD FOR FORMULATION

To calculate the coefficients of the consistent mass matrix, it is necessary first to determine the displacement functions corresponding to unit displacement of each node. The displacement function, $N_i(x)$, is the vertical deflection (from the initial undeformed shape to the deformed shape) of every point on the elbow, when degree of freedom for corresponding node of the pipe elbow is displaced by unity. The vertical deflection is assumed to vary linearly with the value of the nodal displacement, for small deflection in the elastic regions of deformation. The total deflection v(x,t), at a point in the elbow is the sum of the displacement functions obtained from the individual nodal displacements.

$$v(x,t) = N_1(x)v_1(t) + N_2(x)\theta_1(t) + N_3(x)v_2(t) + N_4(x)\theta_2(t)$$

where, vi(t) and i(t) are the vertical and angular deflection of node 'i', at time 't'. Acceleration at a point located 'x' distance from the origin of the elbow at time 't' is given as

$$\frac{\partial^2}{\partial t^2}v(x,t) = N_1(x)\ddot{v}_1(t) + N_2(x)\ddot{\theta}_1(t) + N_3(x)\ddot{v}_2(t) + N_4(x)\ddot{\theta}_2(t)$$

The inertial force developed on an infinitesimal element at a location 'x', due to nodal accelerations, is obtained by the product of the mass of the infinitesimal element 'δm' and the acceleration of the element. When such inertial force is distributed along the beam, unit nodal displacement in each degree of freedom is achieved by applying a force or moment. The external work done by the applied force or moment is equated to the internal work done by the inertial force acting on the beam. Equating the internal work done with the external work done by the nodal forces, the relationship between the acceleration and the applied forces is obtained. The below table provides a clear picture of the virtual work method in derivation of the mass matrix of pipe elbow.

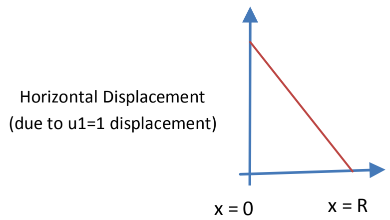
Sl. No	Stage-1 (Acceleration Applied)					Stage-2 (Nodal Displacement applied)						Internal Energy Stored		Work Done
	Imposed set of Nodal Accelerations				Inertial force on Infinitesimal element of mass 'δm' located at distance 'x' from origin due to imposed acceleration	Imposed set of Nodal Displacements				Vertical deflection of each infinitesimal element due to Unit Nodal Deflection	Internal work done by the inertial force acting on 'δm' located at distance 'x' from origin due to imposed displacement	The Internal work done by the inertial force on the entire stretch of elbow during each condition.	Expressed with mass Coefficients	External Work Done
	$\ddot{v}_1$	$\ddot{\theta}_1$	$\ddot{v}_2$	$\ddot{\theta}_2$		v_1	θ_1	v_2	θ_2					
1	$\ddot{v}_1$	0	0	0	$\delta m N_1(x)\ddot{v}_1(t)$	1	0	0	0	$N_1(x)$	$\delta m N_1(x)N_1(x)\ddot{v}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_1(x)N_1(x)\ddot{v}_1(t)dx$	$m'_{11}\ddot{v}_1(t)$	F_{y1}
2	0	$\ddot{\theta}_1$	0	0	$\delta m N_2(x)\ddot{\theta}_1(t)$	1	0	0	0	$N_1(x)$	$\delta m N_2(x)N_1(x)\ddot{\theta}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_1(x)N_2(x)\ddot{\theta}_1(t)dx$	$m'_{12}\ddot{\theta}_1(t)$	
3	0	0	$\ddot{v}_2$	0	$\delta m N_3(x)\ddot{v}_2(t)$	1	0	0	0	$N_1(x)$	$\delta m N_3(x)N_1(x)\ddot{v}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_1(x)N_3(x)\ddot{v}_2(t)dx$	$m'_{13}\ddot{v}_2(t)$	
4	0	0	0	$\ddot{\theta}_2$	$\delta m N_4(x)\ddot{\theta}_2(t)$	1	0	0	0	$N_1(x)$	$\delta m N_4(x)N_1(x)\ddot{\theta}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_1(x)N_4(x)\ddot{\theta}_2(t)dx$	$m'_{14}\ddot{\theta}_2(t)$	
5	$\ddot{v}_1$	0	0	0	$\delta m N_1(x)\ddot{v}_1(t)$	0	1	0	0	$N_2(x)$	$\delta m N_1(x)N_2(x)\ddot{v}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_2(x)N_1(x)\ddot{v}_1(t)dx$	$m'_{21}\ddot{v}_1(t)$	M_{z1}
6	0	$\ddot{\theta}_1$	0	0	$\delta m N_2(x)\ddot{\theta}_1(t)$	0	1	0	0	$N_2(x)$	$\delta m N_2(x)N_2(x)\ddot{\theta}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_2(x)N_2(x)\ddot{\theta}_1(t)dx$	$m'_{22}\ddot{\theta}_1(t)$	
7	0	0	$\ddot{v}_2$	0	$\delta m N_3(x)\ddot{v}_2(t)$	0	1	0	0	$N_2(x)$	$\delta m N_3(x)N_2(x)\ddot{v}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_2(x)N_3(x)\ddot{v}_2(t)dx$	$m'_{23}\ddot{v}_2(t)$	
8	0	0	0	$\ddot{\theta}_2$	$\delta m N_4(x)\ddot{\theta}_2(t)$	0	1	0	0	$N_2(x)$	$\delta m N_4(x)N_2(x)\ddot{\theta}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_2(x)N_4(x)\ddot{\theta}_2(t)dx$	$m'_{24}\ddot{\theta}_2(t)$	
9	$\ddot{v}_1$	0	0	0	$\delta m N_1(x)\ddot{v}_1(t)$	0	0	1	0	$N_3(x)$	$\delta m N_1(x)N_3(x)\ddot{v}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_3(x)N_1(x)\ddot{v}_1(t)dx$	$m'_{31}\ddot{v}_1(t)$	F_{y2}
10	0	$\ddot{\theta}_1$	0	0	$\delta m N_2(x)\ddot{\theta}_1(t)$	0	0	1	0	$N_3(x)$	$\delta m N_2(x)N_3(x)\ddot{\theta}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_3(x)N_2(x)\ddot{\theta}_1(t)dx$	$m'_{32}\ddot{\theta}_1(t)$	
11	0	0	$\ddot{v}_2$	0	$\delta m N_3(x)\ddot{v}_2(t)$	0	0	1	0	$N_3(x)$	$\delta m N_3(x)N_3(x)\ddot{v}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_3(x)N_3(x)\ddot{v}_2(t)dx$	$m'_{33}\ddot{v}_2(t)$	
12	0	0	0	$\ddot{\theta}_2$	$\delta m N_4(x)\ddot{\theta}_2(t)$	0	0	1	0	$N_3(x)$	$\delta m N_4(x)N_3(x)\ddot{\theta}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_3(x)N_4(x)\ddot{\theta}_2(t)dx$	$m'_{34}\ddot{\theta}_2(t)$	
13	$\ddot{v}_1$	0	0	0	$\delta m N_1(x)\ddot{v}_1(t)$	0	0	0	1	$N_4(x)$	$\delta m N_1(x)N_4(x)\ddot{v}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_4(x)N_1(x)\ddot{v}_1(t)dx$	$m'_{41}\ddot{v}_1(t)$	M_{z2}
14	0	$\ddot{\theta}_1$	0	0	$\delta m N_2(x)\ddot{\theta}_1(t)$	0	0	0	1	$N_4(x)$	$\delta m N_2(x)N_4(x)\ddot{\theta}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_4(x)N_2(x)\ddot{\theta}_1(t)dx$	$m'_{42}\ddot{\theta}_1(t)$	
15	0	0	$\ddot{v}_2$	0	$\delta m N_3(x)\ddot{v}_2(t)$	0	0	0	1	$N_4(x)$	$\delta m N_3(x)N_4(x)\ddot{v}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_4(x)N_3(x)\ddot{v}_2(t)dx$	$m'_{43}\ddot{v}_2(t)$	
16	0	0	0	$\ddot{\theta}_2$	$\delta m N_4(x)\ddot{\theta}_2(t)$	0	0	0	1	$N_4(x)$	$\delta m N_4(x)N_4(x)\ddot{\theta}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_4(x)N_4(x)\ddot{\theta}_2(t)dx$	$m'_{44}\ddot{\theta}_2(t)$	

Using Work Energy principle, External Work done by a force to move a unit nodal displacement is equated to the internal work done by the inertial force. Putting in the matrix form

$$\begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} & m'_{13} & m'_{14} \\ m'_{21} & m'_{22} & m'_{23} & m'_{24} \\ m'_{31} & m'_{32} & m'_{33} & m'_{34} \\ m'_{41} & m'_{42} & m'_{43} & m'_{44} \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} = [m'] \{\ddot{x}\}$$

2.2 INCLUSION OF AXIAL EFFECTS

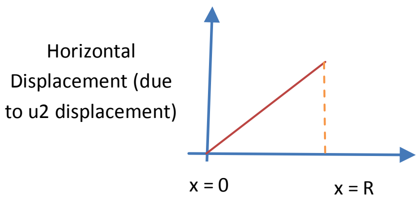
When horizontal unit displacement is applied at node-1, the distance moved by different points in the elbow is shown in the following diagram. The line slopes from unity to zero for points from origin to 'R', when u1=1.



N_5 (x) is the horizontal displacement function, which gives horizontal movement of a point in elbow located 'x' distance from the origin due to unit horizontal displacement at node-1.

The horizontal displacement at a location for imposed displacement at node 1 is given by, $N_5(x) = \left(1 - \frac{x}{R}\right)$

Similarly, when the node -2 is given a unit horizontal displacement, the horizontal displacement distribution is shown in the following diagram.



N_6 (x) is the horizontal displacement function, which gives horizontal movement of a point in elbow located 'x' distance from the origin due to unit horizontal displacement at node-2.

The horizontal displacement at a location for imposed displacement at node 2 is given by, $N_6(x) = \left(\frac{x}{R}\right)$

Net horizontal displacement along x direction as a function of nodal deflections is given as

$$u(x) = N_5(x)\, u1 + N_6(x)\, u_2$$

Stage-1 (Acceleration Applied)					**Stage-2 (Nodal Displacement applied)**		**Internal Energy Stored**	
Imposed set of Nodal Accelerations		Inertial force on Infinitesimal element of mass 'δm' located at distance 'x' from origin due to imposed acceleration	Imposed set of Nodal Displacements		External Work Done (is the applied force itself as displacement is unity)	Internal work done by the inertial force acting on 'δm' located at distance 'x' from origin due to imposed displacement	The Internal work done by the inertial force on the entire stretch of elbow during each condition.	Expressed using mass Coefficients
$\ddot{u}_1$	$\ddot{u}_2$		u_1	u_2				
$\ddot{u}_1$	0	$\delta m N_5(x)\ddot{u}_1(t)$	1	0	$N_5(x)$	$\delta m N_5(x) N_5(x)\ddot{u}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_5(x)N_5(x)\ddot{u}_1(t)dx$	$m''_{11}\ddot{u}_1(t)$
0	$\ddot{u}_2$	$\delta m N_6(x)\ddot{u}_2(t)$	1	0	$N_5(x)$	$\delta m N_6(x) N_5(x)\ddot{u}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_6(x)N_5(x)\ddot{u}_2(t)dx$	$m''_{12}\ddot{u}_2(t)$
$\ddot{u}_1$	0	$\delta m N_5(x)\ddot{u}_1(t)$	0	1	$N_6(x)$	$\delta m N_5(x) N_6(x)\ddot{u}_1(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_5(x)N_6(x)\ddot{u}_1(t)dx$	$m''_{11}\ddot{u}_1(t)$
0	$\ddot{u}_2$	$\delta m N_6(x)\ddot{u}_2(t)$	0	1	$N_6(x)$	$\delta m N_6(x) N_6(x)\ddot{u}_2(t)$	$\frac{\bar{m}\pi}{2}\int_0^R N_6(x)N_6(x)\ddot{u}_2(t)dx$	$m''_{12}\ddot{u}_2(t)$

External work done (with unit displacements) and internal energy are equated and written in Matrix form as below.

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

Including the axial effects (derived in sec 2.3) to the mass matrix coefficients (m'_{ij}) derived in section 2.4, the complete mass matrix for a planar elbow is formulated as given below.

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ M_{z1} \\ F_{x2} \\ F_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} m"_{11} & 0 & 0 & m"_{12} & 0 & 0 \\ 0 & m'_{11} & m'_{12} & 0 & m'_{13} & m'_{14} \\ 0 & m'_{21} & m'_{22} & 0 & m'_{23} & m'_{24} \\ m"_{11} & 0 & 0 & m"_{12} & 0 & 0 \\ 0 & m'_{31} & m'_{32} & 0 & m'_{33} & m'_{34} \\ 0 & m'_{41} & m'_{42} & 0 & m'_{43} & m'_{44} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$

The coefficients are renumbered for the complete mass matrix. The inertial force and the nodal acceleration for a planar elbow are related with the mass matrix as given below.

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ M_{z1} \\ F_{x2} \\ F_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & 0 & m_{14} & 0 & 0 \\ 0 & m_{22} & m_{23} & 0 & m_{25} & m_{26} \\ 0 & m_{32} & m_{33} & 0 & m_{35} & m_{36} \\ m_{41} & 0 & 0 & m_{44} & 0 & 0 \\ 0 & m_{52} & m_{53} & 0 & m_{55} & m_{56} \\ 0 & m_{62} & m_{63} & 0 & m_{65} & m_{66} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$

The inertial forces and nodal accelerations which are related by the mass matrix as given above, need to be determined. To determine the mass coefficients, the shape functions have to be found.
The vertical displacement function is given below.

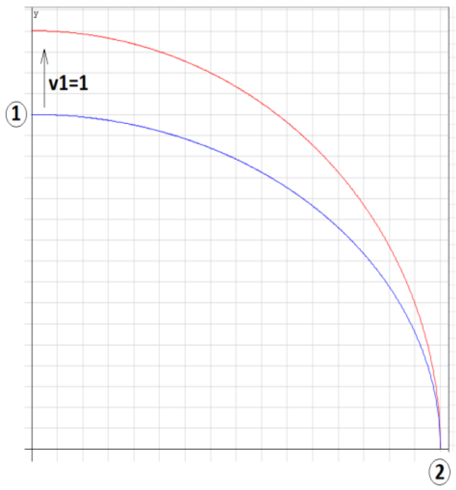
$$v(x,t) = N_1(x)v_1(t) + N_2(x)\theta_1(t) + N_3(x)v_2(t) + N_4(x)\theta_2(t)$$

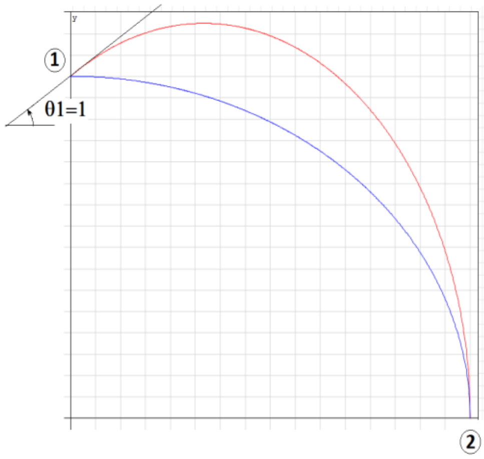
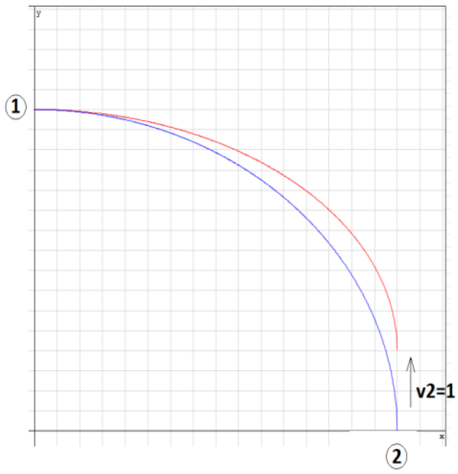
The Horizontal displacement function is given below.

$$u(x,t) = N_5(x)u_1(t) + N_6(x)u_2(t)$$

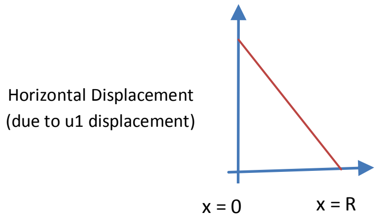
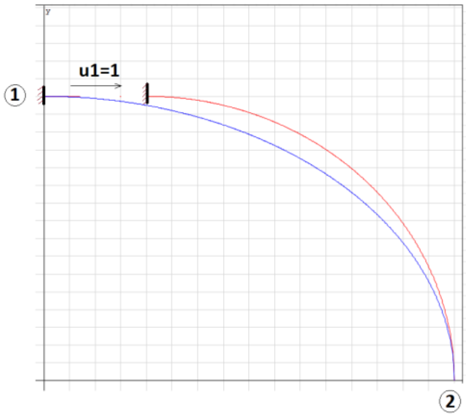
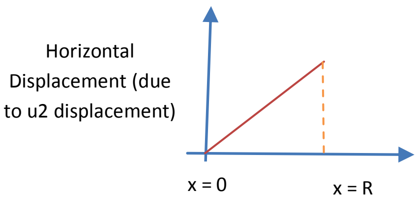
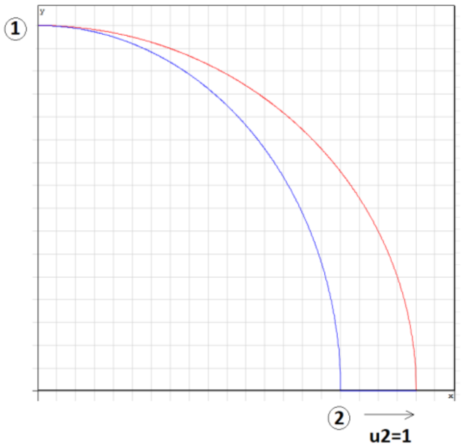
In the section below, the different shape functions will be derived.

2.3 DERIVATION OF SHAPE FUNCTIONS FOR ELBOW AND THEIR PLOT

Shape Function	**Boundary Conditions**	**Plot of Deflected Shape**
$N_1(x)$	The node 1 of pipe elbow is moved vertically up by unit distance. The elbow centerline which was part of a circle, now takes the shape of a parabola, with its major axis as the 'y' axis. The equation of the deflected shape, y(x), is given in the implicit form below. $\frac{x^2}{R^2} + \frac{y^2}{(R+1)^2} = 1$ The ordinate difference between the deflected shape and the original shape, $N_1(x)$ is given by. $N_1(x) = \left((R+1) \times \sqrt{1 - \frac{x^2}{R^2}}\right) - \left(\sqrt{R^2 - x^2}\right)$	

$N_2(x)$	The deflected shape of the element, y(x), when node-1 is rotated anticlockwise by unity, is derived as follows. Boundary Conditions: $y'(x=R) = \infty$; $y'(x=0) = 1$ Satisfying the above slope conditions, $y'(x) = \frac{ax + R}{y}$ $ydy = (ax + R)dx$ $\frac{y^2}{2} = \frac{ax^2}{2} + Rx + C$ Applying boundary condition, y(x=0)=R gives, $C = \frac{R^2}{2}$ Applying boundary condition, y(x=R)=0 gives, $0 = \frac{aR^2}{2} + R^2 + \frac{R^2}{2};\ a = -3$ The ordinate difference between the deflected shape and the original shape, $N_2(x)$ is given below. $N_2(x) = \left(\sqrt{-3x^2 + 2Rx + R^2}\right) - \left(\sqrt{R^2 - x^2}\right)$	
$N_3(x)$	The node 2 of pipe elbow is moved vertically up by unit distance. The elbow centerline which was part of a circle, now takes the shape of a parabola, with major axis as the x axis. The equation of the deflected shape, y(x), is given in the implicit form below. $\frac{x^2}{R^2} + \frac{(y-1)^2}{(R-1)^2} = 1$ The ordinate difference between the deflected shape and the original shape, $N_3(x)$ is given below. $N_3(x) = \left(1 + (R-1) \times \sqrt{1 - \frac{x^2}{R^2}}\right) - \left(\sqrt{R^2 - x^2}\right)$	

$N_4(x)$	The deflected shape of the element, y(x), when node-2 is rotated anticlockwise by unity, is derived as follows.
	Boundary Conditions:
	y(x=0)=R;
	y(x=R)=0;
	$y'(x=0) = 0$;
	$y'(x=R)=1$
	With four boundary conditions, the general equation can be written as
	$y(x) = ax^3 + bx^2 + cx + d$
	The ordinate difference between the deflected shape and the original shape, $N_4(x)$ is given below.
	$N_4(x) = \left(\frac{3}{R^2}x^3 - \frac{4}{R}x^2 + R\right) - \left(\sqrt{R^2 - x^2}\right)$
		

$N_5(x)$	The horizontal displacement at a location for imposed unit displacement at node-1 is given by, $N_5(x) = \left(1 - \frac{x}{R}\right)$ 	
$N_6(x)$	The horizontal displacement at a location for imposed unit displacement at node-2 is given by, $N_6(x) = \left(\frac{x}{R}\right)$ 	

2.4 MASS COEFFICIENTS OF THE MASS MATRIX

The mass matrix is consolidated from the above sections and is given below.

$$M = \overline{m} \times \frac{\pi}{2} \times \begin{bmatrix} \int_0^R N_5N_5dx & 0 & 0 & \int_0^R N_5N_6dx & 0 & 0 \\ 0 & \int_0^R N_1N_1dx & \int_0^R N_1N_2dx & 0 & \int_0^R N_1N_3dx & \int_0^R N_1N_4dx \\ 0 & \int_0^R N_2N_1dx & \int_0^R N_2N_2dx & 0 & \int_0^R N_2N_3dx & \int_0^R N_2N_4dx \\ \int_0^R N_6N_5dx & 0 & 0 & \int_0^R N_6N_6dx & 0 & 0 \\ 0 & \int_0^R N_3N_1dx & \int_0^R N_3N_2dx & 0 & \int_0^R N_3N_3dx & \int_0^R N_3N_4dx \\ 0 & \int_0^R N_4N_1dx & \int_0^R N_4N_2dx & 0 & \int_0^R N_4N_3dx & \int_0^R N_4N_4dx \end{bmatrix}$$

Presenting closed form solutions will be too detailed and hence the mass matrix for different elbows is computed and tabulated below. The mass per unit length, $\bar{m}$ in kg/m, is to be separately calculated depending on pipe schedule and is to be used along with the mass coefficients matrix given below.

TABLE -1: Mass Matrix Coefficients for Standard Pipe Sizes

Sl.No	Elbow Size	$Mass\ Matrix = \bar{m} \times 10^{-2} \times$ Matrix					
1	4"	8.98	0	0	4.49	0	0
		0	17.95	0.68	0	3.2	-1.23
		0	0.68	0.03	0	0.19	-0.07
		4.49	0	0	8.98	0	0
		0	3.2	0.19	0	2.58	-0.47
		0	-1.23	-0.07	0	-0.47	0.15
2	6"	13.22	0	0	6.61	0	0
		0	26.44	1.47	0	4.71	-2.67
		0	1.47	0.1	0	0.41	-0.21
		6.61	0	0	13.22	0	0
		0	4.71	0.41	0	3.8	-1.02
		0	-2.67	-0.21	0	-1.02	0.47
3	8"	17.21	0	0	8.6	0	0
		0	34.42	2.49	0	6.13	-4.52
		0	2.49	0.22	0	0.69	-0.46
		8.6	0	0	17.21	0	0
		0	6.13	0.69	0	4.95	-1.73
		0	-4.52	-0.46	0	-1.73	1.04
4	10"	21.45	0	0	10.72	0	0
		0	42.9	3.87	0	7.64	-7.03
		0	3.87	0.43	0	1.07	-0.89
		10.72	0	0	21.45	0	0
		0	7.64	1.07	0	6.17	-2.69
		0	-7.03	-0.89	0	-2.69	2.01
5	12"	25.44	0	0	12.72	0	0
		0	50.87	5.44	0	9.06	-9.88
		0	5.44	0.72	0	1.51	-1.48
		12.72	0	0	25.44	0	0
		0	9.06	1.51	0	7.32	-3.78
		0	-9.88	-1.48	0	-3.78	3.34
6	14"	21.45	0	0	10.72	0	0
		0	42.9	3.87	0	7.64	-7.03
		0	3.87	0.43	0	1.07	-0.89
		10.72	0	0	21.45	0	0
		0	7.64	1.07	0	6.17	-2.69
		0	-7.03	-0.89	0	-2.69	2.01
7	18"	35.91	0	0	17.95	0	0
		0	71.82	10.83	0	12.79	-19.7

		0	10.83	2.03	0	3.01	-4.16
		17.95	0	0	35.91	0	0
		0	12.79	3.01	0	10.33	-7.54
		0	-19.7	-1.416	0	-7.54	9.41
8	20”	39.9	0	0	19.95	0	0
		0	79.8	13.38	0	14.21	-24.32
		0	13.38	2.78	0	3.72	-5.71
		19.95	0	0	39.9	0	0
		0	14.21	3.72	0	11.48	-9.31
		0	-24.32	-5.71	0	-9.31	12.91
9	24”	47.88	0	0	23.94	0	0
		0	95.76	19.26	0	17.05	-35.02
		0	19.26	4.81	0	5.35	-9.86
		23.94	0	0	47.88	0	0
		0	17.05	5.35	0	13.77	-13.41
		0	-35.02	-9.86	0	-13.41	22.3

To find Mass matrix for a particular pipe elbow, coefficients of the matrix given for each pipe elbow size, has to be multiplied with $\bar{\boldsymbol{m}} \times \mathbf{10^{-2}}$**,** where mass per unit meter, $\bar{\boldsymbol{m}}$ is in kg/m.

III. CONCLUSION

Virtual work method is used to derive the 2D mass matrix of pipe elbow, which can be used for determining the overall mass matrix of the piping system. Mass matrices for different standard pipe elbows are given in this paper and solutions given in simple form, instead of large algebraic equations. The Coordinate transfer method can be used for the elbow in different orientations, similar to the methods used for skewed beam elements. Based on the schedule (thickness) and material of the pipe elbow, the mass per unit length m (kg/m), of the pipe elbow can be found using the tabulated values. Thus, instead of an equivalent pipe length method, a consistent mass matrix of pipe elbow is determined using virtual work method and presented in this paper.

REFERENCES

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