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A New Method for Studying Fractional Fourier Series

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Abstract
In this paper, a new method is used to find the fractional Fourier series of two types of fractional functions. This method is different from the traditional fractional Fourier series method. A new multiplication and fractional analytic functions play important roles in this article. On the other hand, some examples are given to illustrate our results.
Keywords: *New method, Fractional Fourier series, New multiplication, Fractional analytic functions.*

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I. INTRODUCTION

Fractional calculus originated in 1695 and almost at the same time as traditional calculus. Fractional calculus is considered to be a useful tool for understanding and simulating many natural and artificial phenomena. It has developed rapidly in different scientific fields in the past few decades, including not only mathematics and physics, but also engineering, biology, finance, economy and chemistry [1-7]. Great progress can be measured by an increasing number of papers, books and conferences [1-10].

In this article, we introduce a new multiplication of fractional analytic functions and use a new method to find the fractional Fourier series of the following two types of fractional functions.

$$[(q-pr^2)+(pr-qr)cos_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes -1}\ , \tag{1}$$

$$[(pr+qr)sin_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes -1}\ , \tag{2}$$

where $0<\alpha\leq 1$, r,θ,p,q are real numbers, $|r|<1$, and $cos_\alpha(\theta^\alpha)$, $sin_\alpha(\theta^\alpha)$ are α-fractional cosine function and sine function respectively. The new multiplication $\otimes$ of fractional analytic functions is a natural generalization of the multiplication of analytic functions. The method we used is different from the method defined by fractional Fourier series [10]. In addition, we give two examples to illustrate the results we obtained.

II. PRELIMINARIES

First, we define the fractional analytic function.

Definition 2.1 ([11])**:** Suppose that θ,θ_0 , and a_k are real numbers for all k, $\theta_0\in(a,b)$, $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval (θ_0-r,θ_0+r), then $f_\alpha(\theta^\alpha)$ is called α-fractional analytic at θ_0, where r is the radius of convergence about θ_0. Moreover, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, some fractional analytic functions are introduced.

Definition 2.2 ([12, 13])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{3}$$

where α is a real number, $\alpha\geq 0$, and z is a complex number.

Definition 2.3 ([14])**:** Let $0<\alpha\leq 1$, and θ be a real number. $E_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)}$ is called α-fractional exponential function, and the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{4}$$

and

$$sin_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}\ . \tag{5}$$

Proposition 2.4 (fractional Euler's formula)**:** *Let* $0<\alpha\leq 1$, θ *be a real number, then*

$$E_\alpha(i\theta^\alpha)=cos_\alpha(\theta^\alpha)+isin_\alpha(\theta^\alpha)\ . \tag{6}$$

Next, a new multiplication of fractional analytic functions is introduced below.

Definition 2.5 ([11])**:** If $0<\alpha\leq 1$, $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}, \tag{7}$$

$$g_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}. \tag{8}$$

Then we define

$$\begin{aligned}&f_\alpha(\theta^\alpha)\otimes g_\alpha(\theta^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\theta^{k\alpha}.\end{aligned} \tag{9}$$

Definition 2.6: Let $0<\alpha\leq 1$. $\left(f_\alpha(\theta^\alpha)\right)^{\otimes n}=f_\alpha(\theta^\alpha)\otimes\cdots\otimes f_\alpha(\theta^\alpha)$ is called the n-th power of the α-fractional analytic function $f_\alpha(\theta^\alpha)$. If $f_\alpha(\theta^\alpha)\otimes g_\alpha(\theta^\alpha)=1$, then $g_\alpha(\theta^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}$.

Proposition 2.7 ([14])**:** *If* $0<\alpha\leq 1$, *and* c,d *are real numbers, then*

$$E_\alpha(c\theta^\alpha)\otimes E_\alpha(d\theta^\alpha)=E_\alpha((c+d)\theta^\alpha). \tag{10}$$

Notation 2.8: Assume that $z=a+ib$ is a complex number, where $i=\sqrt{-1}$, and a,b are real numbers. Then a, the real part of z, is denoted as $\mathrm{Re}(z)$; b, the imaginary part of z, is denoted as $\mathrm{Im}(z)$.

Proposition 2.9 ([15])**:** *If* $0<\alpha\leq 1$, *then*

$$\left(sin_\alpha(\theta^\alpha)\right)^{\otimes 2}=\frac{1}{2}\left(1-cos_\alpha(2\theta^\alpha)\right), \tag{11}$$

$$\left(cos_\alpha(\theta^\alpha)\right)^{\otimes 2}=\frac{1}{2}\left(1+cos_\alpha(2\theta^\alpha)\right), \tag{12}$$

and

$$sin_\alpha(\theta^\alpha)\otimes cos_\alpha(\theta^\alpha)=\frac{1}{2}sin_\alpha(2\theta^\alpha). \tag{13}$$

III. RESULTS AND EXAMPLES

In order to get the major results, we need the following two lemmas.

Lemma 3.1: *Let* $0<\alpha\leq 1$, r,θ *be real numbers, and* $|r|<1$. *Then*

$$\left(1-rE_\alpha(i\theta^\alpha)\right)^{\otimes -1}=\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha). \tag{14}$$

Proof Since

$$\begin{aligned}&\left(1-rE_\alpha(i\theta^\alpha)\right)\otimes\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha)\\&=\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha)-rE_\alpha(i\theta^\alpha)\otimes\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha)\\&=\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha)-\sum_{n=0}^{\infty}r^{n+1}E_\alpha(i(n+1)\theta^\alpha)\\&=1\,.\end{aligned}$$

It follows that the desired result holds. Q.e.d.

Lemma 3.2: *Suppose that* $0<\alpha\leq 1$, r,θ,p,q *are real numbers, and* $|r|<1$. *Then*

$$\left(prE_\alpha(i\theta^\alpha)+q\right)\otimes\left(1-rE_\alpha(i\theta^\alpha)\right)^{\otimes -1}=q+(p+q)\cdot\sum_{n=1}^{\infty}r^nE_\alpha(in\theta^\alpha). \tag{15}$$

Proof By Lemma 3.1, we have

$$\begin{aligned}&\left(prE_\alpha(i\theta^\alpha)+q\right)\otimes\left(1-rE_\alpha(i\theta^\alpha)\right)^{\otimes -1}\\&=\left(prE_\alpha(i\theta^\alpha)+q\right)\otimes\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha)\\&=p\cdot\sum_{n=0}^{\infty}r^{n+1}E_\alpha(i(n+1)\theta^\alpha)+q\cdot\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha)\\&=-p+(p+q)\cdot\sum_{n=0}^{\infty}r^nE_\alpha(in\theta^\alpha).\\&=q+(p+q)\cdot\sum_{n=1}^{\infty}r^nE_\alpha(in\theta^\alpha)\end{aligned}$$

Q.e.d.

Our main results are as follows.

Theorem 3.3: *If* $0<\alpha\leq 1$, r,θ,p,q *are real numbers, and* $|r|<1$. *Then the* α*-fractional Fourier series*

$$[(q-pr^2)+(pr-qr)cos_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes -1}=q+(p+q)\cdot\sum_{n=1}^{\infty}r^ncos_\alpha(n\theta^\alpha), \tag{16}$$

and

$$[(pr+qr)sin_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes -1}=(p+q)\cdot\sum_{n=1}^{\infty}r^nsin_\alpha(n\theta^\alpha). \tag{17}$$

Proof Since $rE_\alpha(i\theta^\alpha)=rcos_\alpha(\theta^\alpha)+irsin_\alpha(\theta^\alpha)$, it follows that

$$\begin{aligned}&\left(prE_\alpha(i\theta^\alpha)+q\right)\otimes\left(1-rE_\alpha(i\theta^\alpha)\right)^{\otimes -1}\\&=[(prcos_\alpha(\theta^\alpha)+q)+iprsin_\alpha(\theta^\alpha)]\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)\right)-irsin_\alpha(\theta^\alpha)\right]^{\otimes -1}\\&=[(prcos_\alpha(\theta^\alpha)+q)+iprsin_\alpha(\theta^\alpha)]\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)\right)+irsin_\alpha(\theta^\alpha)\right]\\&\quad\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)\right)^{\otimes 2}+\left(rsin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes -1}\\&=\begin{bmatrix}\left[(prcos_\alpha(\theta^\alpha)+q)\otimes\left(1-rcos_\alpha(\theta^\alpha)\right)-pr^2\left(sin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]\\+i\left[(prcos_\alpha(\theta^\alpha)+q)\otimes rsin_\alpha(\theta^\alpha)+prsin_\alpha(\theta^\alpha)\otimes\left(1-rcos_\alpha(\theta^\alpha)\right)\right]\end{bmatrix}\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes -1}.\end{aligned}$$

(18)

Therefore, by Proposition 2.9,

$$\mathrm{Re}\left[(prE_\alpha(i\theta^\alpha)+q)\otimes\left(1-rE_\alpha(i\theta^\alpha)\right)^{\otimes-1}\right]$$
$$=\left[(prcos_\alpha(\theta^\alpha)+q)\otimes\left(1-rcos_\alpha(\theta^\alpha)\right)-pr^2\left(sin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes-1}$$
$$=[(q-pr^2)+(pr-qr)cos_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes-1}. \quad (19)$$

Using Lemma 3.2 yields

$$[(q-pr^2)+(pr-qr)cos_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes-1}$$
$$=q+(p+q)\cdot\mathrm{Re}[\textstyle\sum_{n=1}^{\infty}r^nE_\alpha(in\theta^\alpha)]$$
$$=q+(p+q)\cdot\textstyle\sum_{n=1}^{\infty}r^ncos_\alpha(n\theta^\alpha)\ .$$

On the other hand, since

$$\mathrm{Im}\left[(prE_\alpha(i\theta^\alpha)+q)\otimes\left(1-rE_\alpha(i\theta^\alpha)\right)^{\otimes-1}\right]$$
$$=\left[(prcos_\alpha(\theta^\alpha)+q)\otimes rsin_\alpha(\theta^\alpha)+prsin_\alpha(\theta^\alpha)\otimes\left(1-rcos_\alpha(\theta^\alpha)\right)\right]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes-1}$$
$$=[(pr+qr)sin_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes-1}. \quad (20)$$

Also by Lemma 3.2, we obtain

$$[(pr+qr)sin_\alpha(\theta^\alpha)]\otimes[1-2rcos_\alpha(\theta^\alpha)+r^2]^{\otimes-1}$$
$$=\mathrm{Im}[q+(p+q)\cdot\textstyle\sum_{n=1}^{\infty}r^nE_\alpha(in\theta^\alpha)]$$
$$=(p+q)\cdot\mathrm{Im}[\textstyle\sum_{n=1}^{\infty}r^nE_\alpha(in\theta^\alpha)]$$
$$=(p+q)\cdot\textstyle\sum_{n=1}^{\infty}r^nsin_\alpha(n\theta^\alpha).$$

Q.e.d.

Example 3.4: From Theorem 3.3, we have

$$\left[\frac{17}{9}-\frac{1}{3}cos_\alpha(\theta^\alpha)\right]\otimes\left[1-\frac{2}{3}cos_\alpha(\theta^\alpha)+\frac{1}{9}\right]^{\otimes-1}=2+3\cdot\sum_{n=1}^{\infty}\frac{1}{3^n}cos_\alpha(n\theta^\alpha), \quad (21)$$

and

$$\left[-\frac{2}{5}sin_\alpha(\theta^\alpha)\right]\otimes\left[1+\frac{2}{5}cos_\alpha(\theta^\alpha)+\frac{1}{25}\right]^{\otimes-1}=2\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n}sin_\alpha(n\theta^\alpha). \quad (22)$$

IV. CONCLUSION

As mentioned above, we use a new approach to obtain the fractional Fourier series of two types of fractional functions. This method is different from the traditional method defined by fractional Fourier series. The new multiplication and fractional analytic functions we introduced play important roles in this paper. In fact, the fractional Fourier series is widely used to solve many problems of fractional differential equations. In the future, we will make use of this new method to expand our research to applied mathematics and fractional calculus.

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