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Commutativity of Prime Γ-MA-semirings with None-zero Jordan Left Derivations and Left Derivations on Closed Lie Ideals

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Abstract
Let M be a 2-torsion free and 3-torsion free prime Γ-MA-semiring. Then every non-zero Jordan left derivation $D: M \longrightarrow M$ makes M commutative. Let L be a closed Lie ideal of a 2-torsion free primeΓ-MA-semiring M. Then Jordan left derivations on L are also left derivations .
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I. Introduction

Let$(M,+)$ and $(\Gamma,+)$be commutative semigroups. M is said to be a Γ-semiring if there exists a map $M \times \Gamma \times M \longrightarrow M$that send the triples (x,α,y) to $x\alpha y$ such that

(i) $m\alpha(n+r) = m\alpha n + m\alpha r$,
(ii) $(m+n)\alpha r = m\alpha r + n\alpha r$,
(iii) $m(\alpha+\beta)n = m\alpha n + m\beta n$,
(iv) $(m\alpha n)\beta r = m\alpha(n\beta r)$,

for all $m,n,r\epsilon M$ and $\alpha,\beta\epsilon\Gamma$.

A Γ-semiring M is additively inverse if for every element $m\epsilon M$ there exists a unique element $m'\epsilon M$ such that

(1.1) $m + m^{'} + m = m$ and $m^{'} + m + m^{'} = m'$.

The following identities are valid:

(1.2) $(m\alpha n)^{'} = m^{'}\alpha n = m\alpha n^{'}, (m+n)^{'} = m^{'} + n^{'}, m^{'}n^{'} = mn, (m^{'})^{'} = m, 0^{'} = 0$,
(1.3) $m + n = 0$ impllies $n = m'$ and $m + m^{'} = 0$, for all $m,n\epsilon M$ and $\alpha\epsilon\Gamma$.

The center of aΓ-semiring M is defined as $Z(M) = \{m\epsilon M: m\alpha n = n\alpha m, \forall n\epsilon M, \alpha\epsilon\Gamma\}$.

An additively inverse Γ-semiring M is said to be a Γ-MA-semiring if

(1.4) $(m + m^{'})\epsilon Z(M)$ for all $m\epsilon M$.

The commutator of any elements$m,n\epsilon M$ can be defined as $[m,n]_\alpha = m\alpha n + n^{'}\alpha m$, and $[m,n]_\alpha = 0$ implies $m\alpha n = n\alpha m$.

The following identities in aΓ-MA-semiring M are straightforward:

(1.5) $[m,n]'_\alpha = [m,n^{'}]_\alpha = [m^{'},n]_\alpha, [m^{'},n^{'}]_\alpha = [m,n]_\alpha, [m,n\alpha m]_\alpha = [m,n]_\alpha \alpha m$,
(1.6) Jacobi Identities: $[m\alpha n,r]_\alpha = m\alpha[n,r]_\alpha + [m,r]_\alpha \alpha n$ and $[m,n\alpha r]_\alpha = n\alpha[m,r]_\alpha + [m,n]_\alpha \alpha r$,for all $m,n,r\epsilon M$ and $\alpha\epsilon\Gamma$.

A Γ-MA-semiring M is prime if $m\alpha M\alpha n = 0$ implies $m = 0$ or $n = 0$ for all $m,n\epsilon M$ and $\alpha\epsilon\Gamma$. A Γ-MA-semiring M is n-torsion free for n>1 if n$m = 0$ only for $m = 0$ in M.A subsemigroup L of a Γ-MA-semiring M is a closed Lie ideal M if $L\Gamma M, M\Gamma L, [L,M]_\Gamma, L\Gamma L \subseteq L$. An additive mapping $D: M \longrightarrow M$ is said to be a left derivation if $D(m\alpha n) = m\alpha d(n) + n\alpha d(m)$. D is said to be a Jordan left derivation if

(1.7) $D(m\alpha m) + 2m^{'}\alpha D(m) = 0$,for all $m\epsilon M$ and $\alpha\epsilon\Gamma$.

Y. Ceven [10] studied on Jordan left derivations on completely prime Γ-rings. He proved the that if a Jordan left derivation on a completely prime Γ-ring is non-zero with an assumption, then the Γ-ring is commutative. He also showed that every Jordan left derivation together with an assumption on a completely prime Γ-ring is a left derivation on it. In this paper, he provided an example of Jordan left derivationsonΓ-rings.

Mustafa Asci and Sahin Ceran [8] worked on a nonzero left derivation d on a prime Γ-ring $M$with an ideal U and the center Z of M such that $d(U) \subseteq U$ and $d^2(U) \subseteq Z$for which M is commutative. They also showed that$M$is commutative with the nonzero left derivation d_1 and right derivation d_2 on Msuch that $d_1(U) \subseteq U$and $d_1 d_2(U) \subseteq Z$.
A.C.Paul and Amitabh Kumer Halder [1] investigated the existence of a non-zero Jordan left derivation from a Γ-ring M into a 2-torsionfree and 3-torsionfree leftΓMmodule$X$ that makes M commutative.They also proved that if $X = M$ is a semiprime Γ-ring then the derivation is a mapping from M into its centre and if M is a prime Γ-ring then every Jordan left derivation$d$ on $M$is a left derivation on M.
H. Hedayati and K. P. Shum [5] introduced Γ-semirings and discussed the congruences and ideals of a Γ-semiring, formation of ideals, homomorphisms on Γ-semirings and commutativity of Γ-semirings.
Yaqoub Ahmed and Wieslaw A. Dudek [11] investigated certain conditions for which a left Jordan derivation on an MA-semiring S is a left derivation and S is commutative.
In this study, we generalize the results of Yaqoub Ahmed and Wieslaw A. Dudek [11] in Γ version. We prove that every non-zero Jordan left derivation D on a 2- and 3-torsion free primeΓ-MA-semiring M implies the commutativity of M. We also show that if L is a closed Lie ideal of a 2-torsion free primeΓ-MA-semiring M then every Jordan left derivations on L is also a left derivation on L.

II. Properties of Jordan Left Derivations on Lie Ideals

Lemma 2.1: Let$M$ be an additively inverse Γ-MA-semiring and$D: M \longrightarrow M$ be an additive mapping. Then $D(m^{'}) = D(m)'$ for all $m\epsilon M$**.**
Proof: Since M is additively inverse, for every$m\epsilon M$ there exist a uniquely determined $m'\epsilon M$ such that $m + m^{'} + m = m$. Thisgives$D(m) + D(m^{'}) + D(m) = D(m)$ and $D(m') + D(m) + D(m') = D(m^{'})$ yield $D(m') = D(m)'$ for all $m\epsilon M$.
Lemma 2.2: Let M be a 2-torsion free Γ-MA-semiring and$D: M \longrightarrow M$ be Jordan left derivation. Then
(a) $[m, m]_\alpha \alpha D(m) = 0$, for all $m\epsilon M$ and $\alpha\epsilon\Gamma$.
(b) IfM is prime, then $D(m + m^{'}) = 0$, for all $m\epsilon M$**.**
Proof: (a): We use the consequence of additively inverse: $m + n = 0$ implies $n = m'$ such that $m + m^{'} = 0$ and the definition of Jordan left derivation to get
$2(m + m^{'})\alpha D(m) = 0$ implies $(m + m^{'})\alpha D(m) = 0$ since M is 2-torsion free. This gives $m\alpha(m + m'\alpha Dm=0$ yielding $m,m\alpha\alpha Dm=0$ for all $m\epsilon M$ and $\alpha\epsilon\Gamma$. Finally, by using (1.2), (1.4), we get $m\alpha[n, n]_\alpha \alpha D(m) = 0$, for all $m, n\epsilon M$ and $\alpha\epsilon\Gamma$.
(b): We use (1.3) and (1.7) to get $(m + m^{'})\alpha D(m) = 0$ which implies $m\alpha M\alpha(D(m) + D(m)^{'}) = 0$ by (1.2) and (1.4). Since M is semiprime, $D(m + m^{'}) = 0$, for all $m\epsilon M$ and $\alpha\epsilon\Gamma$.
Lemma 2.3: Let L be a closed Lie ideal of Γ-MA-semiring M and $d: L \longrightarrow L$ be a Jordan left derivation. Then
(a) $d(x\alpha y + y\alpha x) = 2x\alpha d(y) + 2y\alpha d(x)$,
(b) $d(x\alpha y\alpha x) = x\alpha x\alpha d(y) + 3x\alpha y\alpha d(x) + y\alpha x\alpha d(x)^{'}$,
(c) $[x, x]_\alpha \alpha d(y) = 0$,
(d) $d(x\alpha y\alpha z + z\alpha y\alpha x) = (x\alpha z + z\alpha x)\alpha d(y) + 3x\alpha y\alpha d(z) + 3z\alpha y\alpha d(x) + y\alpha x\alpha d(z)^{'} + y\alpha z\alpha d(x)^{'}$,
(e) $[x, y]_\alpha \alpha x\alpha d(x) + x^{'}\alpha[x, y]_\alpha \alpha d(x) = 0$,
(f) $[\boldsymbol{x}, \boldsymbol{y}]_\alpha \alpha(d(x\alpha y) + x\alpha d(y)^{'} + y\alpha d(x)^{'}) = 0$,
for all $x, y, z\epsilon L$ and $\alpha\epsilon\Gamma$.
Proof: (a): By (1.3) and (1.7), we have
$d(x\alpha x) = 2x\alpha d(x)$, (1)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
Replacing x by $x + y$ in the equation (1) and then adding $2x^{'}\alpha d(x) + 2y'\alpha d(y)$ after that using (1.3) and (1.7), we get
$d(x\alpha y + y\alpha x) = 2x\alpha d(y) + 2y\alpha d(x)$, for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
(b):Since$L$ is a closed Lie ideal of M, $(x\alpha y + y\alpha x)\epsilon L$ for all $x, y\epsilon M$ and $\alpha\epsilon\Gamma$. Replacing y by $x\alpha y + y\alpha x$ in (a), we get
$d(x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\alpha x) = 4x\alpha x\alpha d(y) + 6x\alpha y\alpha d(x) + 2y\alpha x\alpha d(x)$,(2)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
Again, by (a) and (1.7), we get
$d(x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\alpha x) = 2x\alpha x\alpha d(y) + 4x\alpha y\alpha d(x) + 2d(x\alpha y\alpha x)$ (3)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
By (1.3) and (1.7), we get $d(x\alpha y + y\alpha x) + d(x\alpha y + y\alpha x)^{'} = 0$ giving
$d(x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\alpha x) + d(x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\alpha x)^{'} = 0$,(4)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.

Since M is 2-torsion free, by equations (2), (3), (4), and ((1.1) with $m = x\alpha x\alpha d(y)$), we get
$x\alpha x\alpha d(y) + y\alpha x\alpha d(x)' + 3x\alpha y\alpha d(x) + d(x\alpha y\alpha x)' = 0,$ (5)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
Now, equation (5) with (1.3) implies (b).
(c): By applying $[x,x]_\alpha \alpha d(y) = x\alpha x\alpha d(y) + x\alpha x\alpha d(y)' = 0$ together with equation (5) and (1.3), we get $[x,x]_\alpha \alpha d(y) = 0$,for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
(d): We replace$x$ and $x + z$ in equation (5) to get
$d\big((\boldsymbol{x} + \boldsymbol{z})\alpha y\alpha(x + z)\big)' + x\alpha x\alpha d(y) + (x\alpha z + z\alpha x)\alpha d(y) + z\alpha z\alpha d(y) + y\alpha x\alpha d(x)' + y\alpha x\alpha d(z)' + y\alpha z\alpha d(x)' + y\alpha z\alpha d(z)' + 3x\alpha y\alpha d(x) + 3z\alpha y\alpha d(x) + 3x\alpha y\alpha d(z) + 3z\alpha y\alpha d(z) = 0,$ (6)
for all $x, y, z\epsilon L$ and $\alpha\epsilon\Gamma$.
Again, by $d\big((\boldsymbol{x} + \boldsymbol{z})\alpha y\alpha(x + z)\big) = d(x\alpha y\alpha) + d(z\alpha y\alpha z) + d(x\alpha y\alpha z + z\alpha y\alpha x)$ and(b),
we get
$d\big((\boldsymbol{x} + \boldsymbol{z})\alpha y\alpha(x + z)\big) = x\alpha x\alpha d(y) + 3x\alpha y\alpha d(x) + y'\alpha x\alpha d(x) + z\alpha z\alpha d(y) + 3z\alpha y\alpha d(z) + y'\alpha z\alpha d(z) + d(x\alpha y\alpha z + z\alpha y\alpha x),$ (7)
for all $x, y, z\epsilon L$ and $\alpha\epsilon\Gamma$.
Using (1.3) in equation (6), we get

$$\alpha x\alpha x\alpha d(y) + x\alpha x\alpha d(y)' = z\alpha z\alpha d(y) + z\alpha z\alpha d(y)' = 3x\alpha y\alpha d(x) + 3x\alpha y\alpha d(x)' = 3z\alpha y\alpha d(z) + 3z\alpha y\alpha d(z)' = y\alpha x\alpha d(x) + y\alpha x\alpha d(x)' = y\alpha z\alpha d(z) + y\alpha z\alpha d(z)' = 0.$$

Now, we use the above equations together with equations (6) and (7) to get
$d(\boldsymbol{x}\alpha y\alpha z + z\alpha y\alpha x)' + (x\alpha z + z\alpha x)\alpha d(y) + 3x\alpha y\alpha d(z) + 3z\alpha y\alpha d(x) + y\alpha x\alpha d(z)' + y\alpha z\alpha d(x)' = 0,$
(8)
for all $x, y, z\epsilon L$ and $\alpha\epsilon\Gamma$, which gives (d).
(e): Since L is a closed Lie ideal, $x\alpha y\epsilon L$ for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
Replacing z by $x\alpha y$ in equation (8), we get
$d((\boldsymbol{x}\alpha y)\alpha(x\alpha y) + x\alpha y\alpha y\alpha x)' + (x\alpha x\alpha y + x\alpha y\alpha x)\alpha d(y) + 3x\alpha y\alpha d(x\alpha y) + 3x\alpha y\alpha y\alpha d(x) + y\alpha x\alpha d(x\alpha y)' + y\alpha x\alpha y\alpha d(x)' = 0,$ (9)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
Applying (1.7) in (b), we get
$d\big((x\alpha y)\alpha(x\alpha y) + x\alpha y\alpha y\alpha x\big) = 2x\alpha y\alpha d(x\alpha y) + 2x\alpha x\alpha y\alpha d(y) + 3x\alpha y\alpha y\alpha d(x) + y\alpha y\alpha x\alpha d(x)',$ which yields with (1.1) and equation (9) as
$x\alpha[x,y]_\alpha \alpha d(y)' + [x,y]_\alpha \alpha d(x\alpha y) + y\alpha[x,y]_\alpha \alpha d(x)' = 0,$ (10)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
Writing $x + y$ for $y$in equation (10) and using (c) and the fact d is a Jordan left derivation, we get

$$2[x,y]_\alpha \alpha x\alpha d(x) + [x,y]_\alpha \alpha d(x\alpha y) + 2x\alpha[x,y]_\alpha \alpha d(x)' + x\alpha[x,y]_\alpha \alpha d(y)' + y\alpha[x,y]_\alpha \alpha d(x)' = 0.$$

Applying the above equation with equation (10) and the fact that M is 2-torsion free, we get $[x,y]_\alpha \alpha x\alpha d(x) + x'\alpha[x,y]_\alpha \alpha d(x) = 0$.
(f): We replace x by $x + y$ in (e) and then use the condition $[x,x]_\alpha \epsilon Z(M)$ and Lemma 2.2 to get

$$[x,y]_\alpha \alpha x\alpha d(y) + [x,y]_\alpha \alpha y\alpha d(x) + [x,y]_\alpha \alpha y\alpha d(y) + x'\alpha[x,y]_\alpha \alpha d(y) + y'\alpha[x,y]_\alpha \alpha d(x) + y'\alpha[x,y]_\alpha \alpha d(y) = 0.$$

Using equation (10) and (1.3), we get
$x'\alpha[x,y]_\alpha \alpha d(y) + y'\alpha[x,y]_\alpha \alpha d(x) = [x,y]_\alpha \alpha d(x\alpha y)'$, and so by (e), (1.2) and (1.5) , we get $[x,y]_\alpha \alpha y\alpha d(y) + y'\alpha[x,y]_\alpha \alpha d(y) = 0$ which gives (f).

Lemma 2.4 Let L be a closed Lie ideal of a 2-torsion free Γ-MA-semiring M and $d: L \longrightarrow L$ be a Jordan left derivation. Then
(a) $[\boldsymbol{x},\boldsymbol{y}]_\alpha \alpha d([x,y]_\alpha) = 0,$
(b) $(\boldsymbol{y}\alpha x\alpha x + 2x\alpha y'\alpha x + x\alpha x\alpha y)\alpha d(y) = 0,$
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.

Proof: (a): By Lemma 2.3(f), we get
$\mathrm{d}[x,y]_\alpha \alpha\big(d(y'\alpha x) + x\alpha d(y) + y\alpha d(x)\big) = 0$ (11)
and
$[x,y]_\alpha \alpha(d(x\alpha y) + x\alpha d(y') + y\alpha d(x')) = 0,$ (12)
for all $x, y\epsilon L$ and $\alpha\epsilon\Gamma$.
We use Lemma 2.2(b) after adding the equations (11) and (12) to get

$$[x,y]_\alpha \alpha d([x,y]_\alpha) = 0.$$

(b): Using the fact L is a closed Lie ideal, (1.7) and (a), we get

$$d(x\alpha(y\alpha x\alpha y) + (y\alpha x\alpha y)\alpha x) + d(x'\alpha y\alpha y\alpha x + y'\alpha x\alpha x\alpha y) = 0.$$

By Lemma 2.3(a), we have

$$2x\alpha d(y\alpha x\alpha y) + 2y\alpha x\alpha y\alpha d(x) + d(x'\alpha y\alpha y\alpha x + y'\alpha x\alpha x\alpha y) = 0.$$

By Lemma 2.3(b) and (1.7) with $2x+3x^{'}=x'$ and $4x^{'}+x=3x'$, we get
$x^{'}\alpha y\alpha y\alpha d(x)+6x\alpha y\alpha x\alpha d(y)+3x\alpha x\alpha y^{'}\alpha d(y)+2y\alpha x\alpha y\alpha d(x)+y\alpha y\alpha x^{'}\alpha d(x)+3y\alpha x\alpha x d(y)^{'}=0$, which yields by (1.2)

$$3(y\alpha x\alpha x+2x\alpha y^{'}\alpha x+x\alpha x\alpha y)\alpha d(y)+(x\alpha y\alpha y+2y\alpha x^{'}\alpha y+y\alpha y\alpha x)\alpha d(x)=0, \quad (13)$$

for all $x,y\epsilon L$ and $\alpha\epsilon\Gamma$.
Applying Lemma 2.3(e) with replacement of x by $x+y$, (1.3) and $x+x^{'}+x=x$, we get

$$\big((x\alpha x\alpha y+2x\alpha y^{'}\alpha x+y\alpha x\alpha x)+(y\alpha y\alpha x^{'}+x^{'}\alpha y\alpha y+2y\alpha x\alpha y)\big)\alpha d(x+y)=0, \quad (14)$$

for all $x,y\epsilon L$ and $\alpha\epsilon\Gamma$.
Lemma 2.3(e) can be written as

$$(y\alpha x\alpha x+x\alpha x\alpha y+2x\alpha y^{'}\alpha x)^{'}\alpha d(x)=0, \quad (15)$$

for all $x,y\epsilon L$ and $\alpha\epsilon\Gamma$.
From (14) and (15), we have

$$(x\alpha x\alpha y+2x\alpha y^{'}\alpha x+y\alpha x\alpha x)\alpha d(y)+(y\alpha y\alpha+x\alpha y\alpha y+2y\alpha x^{'}\alpha y)^{'}\alpha d(x)=0, \quad (16)$$

for all $x,y\epsilon L$ and $\alpha\epsilon\Gamma$.
Adding equations (13) and (16) and then Lemma 2.2(b) and the fact M is 2-torsion free, we get

$$(y\alpha x\alpha x+2x\alpha y^{'}\alpha x+x\alpha x\alpha y)\alpha d(y)=0.$$

Lemma 2.5 Let$L$ be a closed Lie ideal of a 2-torsion free prime Γ-MA-semiring M such that $[L,L]_\Gamma\neq 0$. Then
(a) The ideal generated by $M\Gamma[L,L]_\Gamma\Gamma$M is contained L,
(b) $[[I,M]_\Gamma,I]_\Gamma\neq 0$, where I is an ideal of M contained in L,
(c) L is prime.

Proof: (a): Since L is a closed Lie ideal, $[x,y]_\alpha,[x,m]_\alpha,[x,y\alpha m]_\alpha,y\alpha[x,m]_\alpha,[x,y\alpha m]_\alpha+y^{'}\alpha[x,m]_\alpha\epsilon L$ for all $m\epsilon M$, $x,y\epsilon L$ and $\alpha\epsilon\Gamma$.
Now, using (1.6) and (1.4), we get
$[x,y\alpha m]_\alpha+y^{'}\alpha[x,m]_\alpha=[x,y]_\alpha\alpha m\epsilon\Gamma$.
By (1.1), (1.2), (1.4) and (1.5), we get
$[[x,y]_\alpha\alpha m,n]_\alpha+[x,y]_\alpha\alpha m\alpha n^{'}=n\alpha[y,x]_\alpha\alpha m$,
for all $m,n\epsilon M$, $x,y\epsilon L$ and $\alpha\epsilon\Gamma$. Thus $M\Gamma[L,L]_\Gamma\Gamma\text{M}\subseteq\text{L}$.
If $I=\langle M\Gamma[L,L]_\Gamma\Gamma M\rangle$ is an ideal generated by $M\Gamma[L,L]_\Gamma\Gamma M$ with $[L,L]_\Gamma\neq 0$, then $I\neq 0$. Thus, elements of I are finite sums of elements $m_i\alpha[x_i,y_i]_\alpha\alpha n_i\epsilon M\Gamma[L,L]_\Gamma\Gamma M\subseteq$ L, and so $I\subseteq$ L.
(b): If $[[I,M]_\Gamma,I]_\Gamma=0$ then $[I,M]_\Gamma\subseteq$ Z(M), and so $[x,m]_\alpha\alpha x=[x,m\alpha x]_\alpha$, where I is defined as in the proof of (a). Since $[x,m\alpha x]_\alpha\epsilon Z(M), m^{'}\alpha[x,m]_\alpha\alpha x=[x,m\alpha x]_\alpha\alpha m^{'}$, and so by (1.5), we get $[x,m]_\alpha\alpha[m,x]_\alpha=0$ yielding $[x,m]_\alpha\Gamma\text{M}\Gamma[m,x]_\alpha=0$. Then $[m,x]_\alpha=0$ and in general $[I,M]_\Gamma=0$. Since I is an ideal, $[x,m]_\alpha\alpha i=0$ for all $m\epsilon M$, $x\epsilon L$, $i\epsilon I$ and $\alpha\epsilon\Gamma$. Thus, $[L,M]_\Gamma\Gamma\text{I}=0$, and so $[L,L]_\Gamma\Gamma\text{I}=0$ yielding $[L,L]_\Gamma\Gamma\text{M}\Gamma\text{I}=0$. Since M is prime and $I\neq 0$, we have $[L,L]_\Gamma=0$, which contradicts our supposition. Therefore, $[[I,M]_\Gamma,I]_\Gamma\neq 0$.
(c): Let $L\Gamma m=0$ for some nonzero $m\epsilon M$. Then $[L,M]_\Gamma\Gamma m\subseteq L\Gamma m=0$. Then we have $[x,n]_\alpha\alpha r\alpha m=0$ for all $n,r\epsilon M$, $x\epsilon L$ and $\alpha\epsilon\Gamma$. Then $[L,M]_\Gamma\Gamma M\Gamma m=0$ gives $[L,M]_\Gamma=0$ in particular, $[L,L]_\Gamma=0$, which is a contradiction. Thus, $L\Gamma m\neq 0$ for every nonzero $m\epsilon M$.
We consider I as defined in the proof of (a). If $x\alpha L\alpha y=0$ for some nonzero $x,y\epsilon L$, then by (1.6), we get $x\alpha i\alpha x\alpha m\alpha z\alpha y=0$ for $m\epsilon M$, $z\epsilon L$, $i\epsilon I$, and $\alpha\epsilon\Gamma$. Thus, $x\Gamma I\Gamma x\Gamma M\Gamma L\Gamma y=0$. Since $L\Gamma m\neq 0$ for every nonzero $m\epsilon M$ and M is prime, $x\Gamma I\Gamma x=0$. Since $x\Gamma I\Gamma M\Gamma x\subseteq x\Gamma\text{I}\Gamma x=0$, either $x\Gamma\text{I}=0$ or $x=0$.
Now, $x\Gamma\text{I}=0$ implies $x\Gamma M\Gamma I=0$ implies $x=0$ or $I=0$.
Now, $I=0$ implies $M\Gamma[L,L]_\Gamma\Gamma M=0$ implies $M\Gamma[L,L]_\Gamma=0$. Thus, $[L,L]_\Gamma\Gamma M\Gamma[L,L]_\Gamma=0$, and so $[L,L]_\Gamma=0$. This contradicts the assumption. So $x=0$. Therefore L is prime.

III. Commutativity of torsion free prime Γ-MA-semirings with Jordan left derivations and closed Lie ideals with left derivations

Lemma 3.1: Let$M$ be a 2-torsion free Γ-MA-semiring and $D: M\longrightarrow M$ be a Jordan left derivation. Then elements of the form $[m,[m,n]_\alpha]_\alpha$ for all $m,n\epsilon M$ and $\alpha\epsilon\Gamma$ such that $D(m)\neq 0$ is nilpotent of index 2.
Proof: Suppose $D(m)\neq 0$ for seme $m\epsilon M$. Then by Lemma 2.3(e) and (1.2), we have $[m,[m,n]_\alpha]_\alpha\alpha D(m)=0$, (17)
for all $n\epsilon M$ and $\alpha\epsilon\Gamma$.
Replacing $n=r\alpha s$ in (17) and then using (1.6) with right Jacobi identity and again by (17)

$$([m,[m,r]_\alpha]_\alpha\alpha r+2[m,r]_\alpha\alpha[m,s]_\alpha)\alpha d(m)=0$$

We write $[m,s\alpha t]_\alpha$ for s in the above equation and use the equation (17) to get
$[m,[m,r]_\alpha]_\alpha\alpha[m,s\alpha t]_\alpha\alpha d(m)=0$, for all $r,s,t\epsilon M$ and $\alpha\epsilon\Gamma$.
By equation (16), we have

$$([m,[m,r]_\alpha]_\alpha\alpha s\alpha[m,t]_\alpha+[m,[m,s]_\alpha]_\alpha\alpha[m,s]_\alpha\alpha t)\alpha d(m)=0, \quad (18)$$

for all $r,s,t\epsilon M$ and $\alpha\epsilon\Gamma$.

Now using $t = [m,t]_\alpha$ in equation (17) and then use it in equation (18) after putting $s = [m,s]_\alpha$, we get $[m,[m,r]_\alpha]_\alpha \alpha [m,[m,s]_\alpha]_\alpha \alpha t\alpha d(m) = 0$, which is true for all $m,r,s,t\epsilon M$ and $\alpha\epsilon\Gamma$, and so $(([m,[m,r]_\alpha]_\alpha \alpha)^2 [m,[m,\ r]_\alpha]_\alpha)\Gamma M\Gamma D(m) = 0$. Since M is prime and $D(m)) \neq 0$, $(([m,[m,r]_\alpha]_\alpha \alpha)^2 [m,[m,\ r]_\alpha]_\alpha) = 0$ for all $r\epsilon M$ and $\alpha\epsilon\Gamma$. Replacing r by n, we have $(([m,[m,n]_\alpha]_\alpha \alpha)^2 [m,[m,\ n]_\alpha]_\alpha) = 0$, and so $[m,[m,\ n]_\alpha]_\alpha$ is nilpotent of index 2.

Lemma 3.2: Let M be a 2-torsion free and 3-torsion freeΓ-MA-semiring and $D: M \longrightarrow M$ be a Jordan left derivation. Then $D(m) = 0$ for all$m\epsilon M$ such that $(m\alpha)^2 m = 0$.

Proof: Suppose that$(m\alpha)^2 m = m\alpha m = 0$. Then $D((m\alpha)^2 m) = D(m\alpha m) = 2m\alpha D(m) = 0$. Since M is 2-torsion free, $2m\alpha D(m) = 0$, and by Lemma 2.3(b), we get

$D(m\alpha(r\alpha m\alpha s + s\alpha m\alpha r)\alpha m) = 3m\alpha r\alpha m\alpha s\alpha D(m) + 3m\alpha s\alpha m\alpha r\alpha D(m)$,(19)

for all $m,r,s\epsilon M$ and $\alpha\epsilon\Gamma$.

Again, $D(m\alpha s\alpha m) = 3m\alpha s\alpha D(m)$. Then by Lemma 2.3(b) and (d), we get

$D(m\alpha(r\alpha m\alpha s + s\alpha m\alpha r)\alpha m) = 9m\alpha r\alpha m\alpha s\alpha D(m) + 3m\alpha s\alpha m\alpha r\alpha D(m)$,(20)

for all $m,r,s\epsilon M$ and $\alpha\epsilon\Gamma$.

By equations (19) and (20), we have

$$3[m\alpha r, m\alpha s]_\alpha \alpha(D(m) + D(m)') + 6m\alpha r\alpha m\alpha s\alpha D(m)' = 0.$$

Since M is 2- and 3-torsion free, by Lemma 2.2(b), we get $m\alpha r\alpha m\alpha s\alpha D(m) = 0$ for all $r,s\epsilon M$ and $\alpha\epsilon\Gamma$, and so $m\Gamma M\Gamma(m\Gamma M\Gamma D(m)) = 0$. Since M is prime and $m \neq 0$, we have $m\Gamma M\Gamma D(m) = 0$ implies $D(m) = 0$.

Theorem 3.3: Let M be 2-torsion free and 3-torsion free primeΓ-MA-semiring and $D: M \longrightarrow M$ be a non-zero Jordan left derivation. Then M is commutative.

Proof: Let $D: M \longrightarrow M$ be a non-zero Jordan left derivation such that $D(m) \neq 0$ for $m(\neq 0)\epsilon M$. Then by Lemma 3.1 and Lemma 3.2,

$$D(m\alpha m\alpha r + r\alpha m\alpha m) + 2D(m\alpha r\alpha m)' = 0,$$

for all $r\epsilon M$ and $\alpha\epsilon\Gamma$.

By Lemma 2.3(a) and (b) and (1.3), we get

$6(r\alpha m + m'\alpha r)\alpha D(m) = 0$,and since M is 2- and 3-torsion free, $[r,m]_\alpha \alpha D(m) = 0$, (21)

for all $m,r\epsilon M$ and $\alpha\epsilon\Gamma$.

Writing r by $s\alpha r$ in equation (21) and using first Jacobi identity, we get $[s,m]_\alpha \alpha r\alpha D(m) = 0$ and so $[s,m]_\alpha = 0$, for all $s\epsilon M$ and $\alpha\epsilon\Gamma$ and so $m\epsilon Z(M)$.

Now, we consider

$$P_1 = \{m\epsilon M: D(m) \neq 0\} \subseteq \mathrm{Z(M)}$$

and $P_2 = \{m\epsilon M: D(m) = 0\}$.

Consider an element $m_1 + m_2$ such that $m_1\epsilon P_1$ and $m_2\epsilon P_2$. Suppose $(m_1 + m_2)\epsilon P_2$. Then $D(m_1 + m_2) = D(m_1) + D(m_2) = D(m_1) = 0$, a contradiction. So, $(m_1 + m_2)\epsilon P_1$, and hence $[s,m_2]_\alpha = [s,m_1 + m_2]_\alpha = 0$ for each $s\epsilon M$ and $\alpha\epsilon\Gamma$ implies$P_2 \subseteq \mathrm{Z(M)}$. Therefore, M is commutative.

Corollary 3.4: LetM be a 2-torsion free and 3-torsion free non-commutative Γ-MA-semiring and $D: M \longrightarrow M$ be a Jordan left derivation. Then $D = 0$.

Theorem 3.5: Let$L$ be a closed Lie ideal of a 2-torsion free primeΓ-MA-semiring $M$and $d: L \longrightarrow L$ be a Jordan left derivation. Then d is left derivation on L.

Proof: Suppose$L$ is a closed Lie ideal of M. Let $[L,L]_\Gamma = 0$. Then$x\alpha y = y\alpha x$, for all $x,y\epsilon L$ and $\alpha\epsilon\Gamma$ and so

$d(x\alpha x) + 2x'\alpha d(x) = 0$, (22)

for all$x\epsilon L$ and $\alpha\epsilon\Gamma$.

Replacing x by $x + y$ in equation (22), we get

$$2d(x\alpha y) + 2x'\alpha d(y) + 2y'\alpha d(x) = 0.$$

Since M is 2-torsion free, $d(x\alpha y) + x'\alpha d(y) + y'\alpha d(x) = 0$. By (1.3), we get

$$d(x\alpha y) = x\alpha d(y) + y\alpha d(x).$$

Suppose $[L,L]_\Gamma \neq 0$. Then by Lemma 2.3(e), we get $(x\alpha x\alpha y + 2x\alpha y'\alpha x + y\alpha x\alpha x)\alpha d(x) = 0$, for all $x,y\epsilon L$ and $\alpha\epsilon\Gamma$.

Putting $x = [w,z]_\alpha$ and then using Lemma 2.4(a) with $[x,y]_\alpha \alpha d([x,y]_\alpha) = 0$, we get $[w,z]_\alpha \alpha [w,z]_\alpha \alpha y\alpha d([w,z]_\alpha) = 0$ for all $x,y,z,w\epsilon L$ and $\alpha\epsilon\Gamma$. This implies that $[w,z]_\alpha \alpha [w,z]_\alpha \Gamma L\Gamma d([w,z]_\alpha) = 0$, and so by Lemma 2.5, we get either $[w,z]_\alpha \alpha [w,z]_\alpha = 0$ or $([w,z]_\alpha) = 0$. If $d([w,z]_\alpha) = 0$, then d is a left derivation. If $[w,z]_\alpha \alpha [w,z]_\alpha = 0$, then by Lemma 2.4(b) with $y = [w,z]_\alpha$ and then using Lemma 2.4(a), we have

$(2x\alpha[w,z]'_\alpha \alpha x + [w,z]_\alpha \alpha x\alpha x)\alpha d([w,z]_\alpha) = 0$, (23)

$x,z,w\epsilon L$ and $\alpha\epsilon\Gamma$.

Replacing x by $x + y$ in equation (23) and then by Lemma 2.4(a) after replacing x by $l\alpha[w,z]_\alpha$ and $[w,z]_\alpha \alpha [w,z]_\alpha = 0$, we have$[w,z]_\alpha \alpha l\alpha [w,z]_\alpha \alpha y\alpha d([w,z]_\alpha) = 0$ all $l,y\epsilon L$ and $\alpha\epsilon\Gamma$. Thus $[w,z]_\alpha \alpha L\alpha [w,z]_\alpha \alpha L\alpha d([w,z]_\alpha) = 0$ which implies either $[w,z]_\alpha \alpha L\alpha [w,z]_\alpha = 0$ or $d([w,z]_\alpha) = 0$. If

$d([w,z]_\alpha) = 0$, then d is a left derivation. If $[w,z]_\alpha \alpha L \alpha [w,z]_\alpha = 0$, then by Lemma 2.5(c), we get $[w,z]_\alpha = 0$. Thus $w\alpha z = z\alpha w$ and so d is a left derivation on L.

Corollary 3.6: Let M be a 2-torsion free primeΓ-MA-semiring. Then every Jordan left derivation D on M is a left derivation on D.

IV. Discussion

We use non-zero Jordan left derivations on 2-torsion free and 3-torsion free prime Γ-MA-semirings M to show the commutativity of M whereasA. C. Paul and Amitabh Kumer Halder [1] use non-zero Jordan left derivations from Γ-rings M into 2-torsionfree and 3-torsionfree left ΓMmodule X to prove commutativity of$M$. Also, we investigate left derivations on closed Lie ideals L of 2-torsion free prime Γ-MA-semiring M whereas A. C. Paul and Amitabh Kumer Halder [1] studyleft derivations on M under the condition that $X = M$is a semiprime.

V. Conclusion

Non-zero Jordan left derivations$D$onn-torsion free prime Γ-MA-semirings Mguarantee the commutativity of M for n=2, 3. Closed Lie ideals$L$ ofn-torsion free primeΓ-MA-semiring M are responsible for transformingJordan left derivations on $L$to left derivations for n=2.

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