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Estimating Shiftunder Linex Loss in Exponentiated Inverted Weibull Distribution

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Abstract
In this paper we have obtained Bayes estimatesshifts in sequence and parameters of the Exponentiated Inverted Weibull Distribution.We have developed a wide-ranging theory to estimate the shifts in the mean of the sequence of purposeful observations of Exponentiated Inverted Weibull Distribution.The Bayes estimates shifts in sequence and parameters of the Exponentiated Inverted Weibull Distribution m are derived for asymmetricLinex loss function with natural conjugate inverted Gamma prior distribution. The theoretical outputs arecompared by a numerical study which demonstrates the performance of the estimator in determinate samples.
Keywords: *Shifts, Exponentiated Inverted Weibull Distribution, Bayesian Method, Natural Conjugate Inverted Gamma Prior, Linex Loss Function.*

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I. Introduction

Bayesian decision theory provides a unified and intuitively appealing approach to drawing inferences from observations and making rational, informed decisions. Bayesians view statistical inference as a problem in *belief dynamics*, of using evidence about a phenomenon to revise and update knowledge about it. Bayesian statistics is a scientifically justifiable way to integrate informed expert judgment with empirical data. For a Bayesian, statistical inference cannot be treated entirely independently of the context of the decisions that will be made on the basis of the inferences. In recent years, Bayesian methods have become increasingly common in a variety of disciplines that rely heavily on data. This course introduces students to Bayesian theory and methodology, including modern computational methods for Bayesian inference.

Statistical decision theory deals with situations where decisions have to be made under a state of uncertainty, and its goal is to provide a rational frame work for dealing with such situations. The Bayesian approach is a particular way of formulating and dealing with statistical decision problems. More specifically, it offers a method of formulizing a priori beliefs and of combining them with the available observations, with the goal of allowing a rational derivation of optimal decision criteria. Soin decision theory and estimation theory, a Bayes estimator is an estimator or decision rule that maximizes the posterior expected value of a utility function or minimizes the posterior expected value of a loss function also called posterior expected loss.

1.1 Loss Function

Let δ be an unknown parameter of some distribution $f(x|\delta)$ and suppose we estimate δ by some statistic$\hat{\delta}$. Let $L(\hat{\delta},\delta)$ represent the loss incurred when the true value of the parameter is δ and we are estimating δ by the statistic$\hat{\delta}$.

The most widely used symmetric loss function in estimation problems is quadratic loss function given as $L(\hat{\delta},\delta) = k(\hat{\delta}-\delta)^2$ where $\hat{\delta}$ is the estimate of δ, the loss function is called quadratic weighed loss function if k=1, we have

$L(\hat{\delta},\delta) = (\hat{\delta}-\delta)^2$known as squared error loss function (SELF).

Linex Loss: A symmetric loss function assumes that positive and negative error are equally serious. However, in some estimation problems such an assumption may be inappropriate. Cannfield (1970) points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Over estimation of reliability function or average lifetime is usually much more serious than under estimation of reliability function or mean failure time.

Also an underestimate of the failure rate results in more serious consequences than an overestimation of the failure rate. This led to the statistician to think about asymmetrical loss function which have been proposed in statistical literature. Ferguson (1967), Zellner & Geisel (1968), Aitchision& Dunsmore (1975) and Berger (1980) have considered the linear asymmetric loss function. Varian (1975) introduced the following convex loss function known as LINEX. (Linear Exponential) Loss Function i.e. given as;

$$L(\Delta) = be^{a\Delta} - c\Delta - b\,;\, a, c \neq 0, b > 0 \quad (1.2.1)$$

Where $\Delta = \hat{\theta} - \theta$. It is clear that L(0) = 0 and the minimum occurs when ab=c, therefore , $L(\Delta)$ can be written as

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1\,], a \neq 0, b > 0 \quad (1.2.2)$$

Where a and b are the parameters of the loss function may be defined as shape and scale respectively. The loss function has been considered by Zellner (1986), Basu and Ebrahimi (1991) considered the $L(\Delta)$ as

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1\,],\ a \neq 0, b > 0 \quad (1.2.3)$$

Where, $\Delta = \frac{\hat{\theta}}{\theta} - 1$

1.3 Shift in Sequence

Physical systems manufacturing the items are often subject to random fluctuations. It may happen that at some point of time instability in the sequence of lifetimes is observed. Such observed point is known as Change or Shift point inference problem. Such Change or Shift point inference problem is useful in statistical quality control to study the Change or Shifting in process mean, Linear time series models, and models related to econometrics. The monographs, Broemeling and Tsurmi (1987) on structural changes and survey by Zack (1981) are useful references. Bayesian approach may play an important role in the study of such Change or Shift point problem and has been often proposed as a valid alternative in classical estimation procedure. A variety of Change or Shift point problems have studied in Bayesian frame work by many authors like Zellner (1986), Calabria and Pulcini(1994) and Jani and Pandya (1999).
The various statistical models in this chapter are considered are as

1.4 Prior Distribution: Natural Conjugate Prior (NCP)

In frequentist framework, sufficient statistic plays an important role in Bayesian inference in constructing a family of prior distributions known as Natural Conjugate Prior (NCP). The family of prior distributions $\xi(\alpha),\ \alpha \in \Omega$, is called a natural conjugate family if the corresponding posterior distribution belongs to the same family as $\xi(\alpha)$. De Groot (1970) has outlined a simple and elegant method of constructing a conjugate prior for a family of distributions $f(x|\alpha)$ which admits a sufficient statistic. De Groot (1970) and Raffia &Schlaifer (1961) provide proof that when a sufficient statistics exist a family of conjugate prior distributions exists.

The most widely used prior distribution of α is the inverted Gamma distribution with the parameters 'a' and 'b' (> 0) with p.d.f. given by

$$g(\alpha) = \begin{cases} \frac{b^a}{\Gamma a} \alpha^{-(\alpha+1)} e^{-b/\alpha}\,; & \alpha > 0\,, (a, b) > 0 \\ 0 & , \text{otherwise} \end{cases} \quad (1.4.1)$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted Gamma distribution is conjugate prior of α Raffia & Schlaifer (1961), Bhattacharya (1967) and others have found that the inverted Gamma can also be used for practical reliability applications.

1.5 Exponentiated Inverted Weibull Distribution

The Inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates.Khan et al.(2008) explained the flexibility of the three parameters inverted Weibull distribution and its interested properties. Exponentiated (generalized) Inverted Weibull Distribution is a generalization to the Inverted Weibull distribution through adding a new shape parameter $\lambda \in \mathcal{R}^+$ by exponentiation to distribution function F, the new distribution function F^{λ}.Al-Hussaini et al.(2010) explained that the cumulative distribution function is flexible to monotone and non-monotone failure rates. Mudholkar et al(1995) introduced the Exponentiated Weibull Distribution as generalization of the standard Weibull Distribution, that applied the new distribution as a suitable model to the bus-motor failure time data. Nasar et al.(2003) reviewed the Exponentiated Weibull Distribution with new measures. Nadarajah et al(2005) discussed in details the moments of the Exponentiated Weibull distribution. Mudholkar et at(1995) applied the Exponentiated Weibull distribution to the flood data with some properties
The two parameter EIW distribution has the following probability density function

$$f(x) = \alpha\beta x^{-(\beta+1)} (e^{-\alpha x})^{-\beta}; \quad x > 0, (\beta > 0, \alpha > 0) \quad (1.5.1)$$

And the distribution function

$F(x) = \left(e^{-x^{-\beta}}\right)^{\alpha}$; $x > 0$(1.5.2)

Also, the reliability function of the EIW distribution with two shape parameters α and β are given by

$\boldsymbol{R(t) = 1 - \left(e^{-t^{-\beta}}\right)^{\alpha}}$; $\boldsymbol{t} > 0$(1.5.3)

1.6 Bayesian Estimation of Change point in Exponentiated Inverted Weibull Distribution under Linex Loss Function (LLF)

A sequence of independent lifetimes $x_{1}, x_{2,} \ldots . x_{k,} x_{(k+1)}, \ldots \ldots x_{l} (l \geq 3)$ were observed from Exponentiated Inverted Weibull Distribution with parameter β, α . But it was found that there was a change in the system at some point of time 'm' and it is reflected in the sequence after 'x_l' whichresults change in a sequence as well as parameter value. The Bayes estimate of α and 'k'are derived for symmetric and asymmetric loss function under inverted gamma prior as natural conjugate prior.

1.6.1 Likelihood, Prior, Posterior and Marginal

Let $x_1, \ldots \ldots \ldots \ldots . x_l$, $(l \geq 3)$ be a sequence of observed discrete life times. First let observations $x_1, \ldots \ldots . x_l$ have come from Exponentiated Inverted Weibull Distribution with probability density function as

$$\mathrm{f}(\mathrm{x}, \beta, \alpha) = \alpha\beta x^{-(\beta+1)}(e^{-\alpha x})^{-\beta}; \qquad (\mathrm{x}, \beta, \alpha > 0)(1.6.1.1)$$

Let 'k' is change point in the observation which breaks the distribution in two sequences as $(x_1, x_2, \ldots \ldots \ldots . . x_k)$ & $(x_{k+1}, x_{k+2}, \ldots \ldots \ldots \ldots x_l)$

The probability density function of the above sequences are

$$f_1(x) = \alpha_1\beta_1 x^{-(\beta_1+1)}(e^{-\alpha_1 x})^{-\beta_1}(1.6.1.2)$$

Where$x_1, x_2, \ldots \ldots \ldots x_k, \alpha_1, \beta_1 > o$

$$f_2(x) = \alpha_2\beta_2 x^{-(\beta_2+1)}(e^{-\alpha_2 x})^{-\beta_2}(1.6.1.3) \qquad x_{(k+1)}, \ldots \ldots x_l, \alpha_2, \beta_2 > 0$$

The likelihood functions of probability density function of the sequence are

$$L_1(x, \alpha_{1,}\beta_1) = \prod_{j=1}^{k} f(x_{j\ ,}\ \alpha_{1,}\beta_1)$$

$$L_1(x, \alpha_{1,}\beta_1) = {\alpha_1}^k {\beta_1}^k \prod_{j=1}^{k} {x_j}^{-(\beta_1+1)} e^{-\alpha_1 \sum_{j=1}^{k} {x_j}^{-\beta_1}}$$

$$L_1(x, \alpha_{1,}\beta_1) = (\alpha_1\beta_1)^k U_1 e^{-\alpha_1 T_{2k}}(1.6.1.4)$$

Where $U_1 = \prod_{j=1}^{k} {x_j}^{-(\beta_1+1)}$

$$T_{2k} = \sum_{j=1}^{k} {x_j}^{-\beta_1}$$

$$L_2(x, \alpha_{2,}\beta_2) = \prod_{j=k+1}^{l} f(x_{j\ ,}\ \alpha_{2,}\beta_2)$$

$$L_2(x, \alpha_{2,}\beta_2) = {\alpha_2}^{l-k} {\beta_2}^{l-k} \prod_{j=k+1}^{l} {x_j}^{-(\beta_2+1)} e^{-\alpha_2 \sum_{j=1}^{k} {x_j}^{-\beta_2}}$$

$$L_2(x, \alpha_{2,}\beta_2) = (\alpha_2\beta_2)^{l-k} U_2 e^{-\alpha_2(T_{2l}-T_{2k})}(1.6.1.5)$$

Where $U_2 = \prod_{j=k+1}^{l} {x_j}^{-(\beta_2+1)}$

$$T_{2l} - T_{2k} = \sum_{j=k+1}^{l} {x_j}^{-\beta_2}$$

And the joint Likelihood function is given by

$\mathrm{L}(\alpha_1, \alpha_2 | \underline{x}) \propto (\alpha_1\beta_1)^k U_1 e^{-\alpha_1 T_{2k}} (\alpha_2\beta_2)^{l-k} U_2 e^{-\alpha_2(T_{2l}-T_{2k})}(1.6.1.6)$

Suppose the marginal prior distributions 0f α_1, α_2 are natural conjugate prior

$$\pi_1(\alpha_1, \underline{\mathrm{x}}) = \frac{b_1^{a_1}}{\Gamma a_1} \alpha_1^{(a_1-1)} e^{-b_1\alpha_1}; \qquad a_{1,} b_1 > 0, \alpha_1 > 0(1.6.1.7)\pi_2(\alpha_2, \underline{\mathrm{x}}) = \frac{b_2^{a_2}}{\Gamma a_2} \alpha_2^{(a_2-1)} e^{-b_2\alpha_2}; \quad a_2, b_2 > 0, \alpha_2 > 0(1.6.1.8)$$

The joint prior distribution of α_1, α_2 and change point 'k' is

$$\pi(\alpha_1,\alpha_2,k) \propto \frac{b_1^{a_1}}{\Gamma a_1}\frac{b_2^{a_2}}{\Gamma a_2}\alpha_1^{(a_1-1)}e^{-b_1\alpha_1}\alpha_2^{(a_2-1)}e^{-b_2\alpha_2} \quad (1.6.1.9)$$

where $\alpha_1,\alpha_2 > 0\ \&\ k = 1,2,\dots\dots(l-1)$

The joint posterior density of α_1,α_2 and k say $\pi(\alpha_1,\alpha_2,k/\underline{x})$ is obtained by using equations (1.6.1.6)&(1.6.1.9)

$$\rho(\alpha_1,\alpha_2,k|\underline{x}) = \frac{\mathrm{L}(\alpha_1,\alpha_2/\underline{x})\pi(\alpha_1,\alpha_2,k)}{\sum_k \iint_{\alpha_1,\alpha_2}\mathrm{L}(\alpha_1,\alpha_2/\underline{x})\pi(\alpha_1,\alpha_2,k)d\alpha_1,d\alpha_2} \quad (1.6.1.10)\rho(\alpha_1,\alpha_2,k|\underline{x})$$

$$= \frac{\alpha_1{}^{(k+a_1-1)}e^{-\alpha_1(T_{2k}+b_1)}\alpha_2{}^{(l-k+a_2-1)}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}}{\sum_m\int_0^\infty e^{-\alpha_1(T_{2k}+b_1)}\alpha_1{}^{(k+a_1-1)}\,d\alpha_1\int_0^\infty \alpha_2{}^{(l-k+a_2-1)}e^{-\alpha_2(T_{2k}-T_{2k}+b_2)}\,d\alpha_2}$$

Assuming $\alpha_1(T_{2k}+b_1) = x\ \ \&\alpha_2(T_{2l}-T_{2k}+b_2) = y$

$$\alpha_1 = \frac{x}{(T_{2k}+b_1)}\&\alpha_2 = \frac{y}{T_{2l}-T_{2k}+b_2}$$

$$d\alpha_1 = \frac{dx}{(T_{2k}+b_1)}\&\ \mathrm{d}\alpha_2 = \frac{dy}{T_{2l}-T_{2k}+b_2}$$

$$\rho(\alpha_1,\alpha_2,k|\underline{x}) = \frac{\alpha_1{}^{(k+a_1-1)}e^{-\alpha_1(T_{2k}+b_1)}\alpha_2{}^{(l-k+a_2-1)}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}}{\sum_k\int_0^\infty e^{-x}\frac{x^{(k+a_1-1)}}{(T_{2k}+b_1)^{(k+a_1-1)}}\frac{dx}{(T_{2k}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{y}}\frac{\mathrm{y}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2-1)}}\frac{dy}{(T_{2l}-T_{2k}+b_2)}}\rho(\alpha_1,\alpha_2,k|\underline{x})$$

$$= \frac{e^{-\alpha_1(T_{2k}+b_1)}\alpha_1{}^{(\alpha+a_1-1)}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}\alpha_2{}^{(l-k+a_2-1)}}{\sum_k\frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}$$

$$\rho(\alpha_1,\alpha_2,m|\underline{x}) = \frac{e^{-\alpha_1(T_{2k}+b_1)}\theta_1{}^{(k+a_1-1)}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}\alpha_2{}^{(l-k+a_2-1)}}{\xi(a_{1,}a_{2,}b_{1,}b_{2,}k,l)} \quad (1.6.1.11)$$

Where $\xi(a_{1,}a_{2,}b_{1,}b_{2,}k,l) = \sum_{k=1}^{l-1}\left[\frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{k+a_1}}\frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}\right]$

The Marginal posterior distribution of change point 'm' using the equations (1.6.1.6), (1.6.1.7)&(1.6.1.8)

$$\rho(k|\underline{x}) = \frac{\mathrm{L}(\alpha_1,\alpha_2/\underline{x})\,\pi(\alpha_1)\,\pi(\alpha_2)}{\sum_m \mathrm{L}(\alpha_1,\alpha_2/\underline{x})\,\pi(\alpha_1)\,\pi(\alpha_2)} \quad (1.6.1.12)$$

On solving which gives

$$\rho(k|\underline{x}) = \frac{\alpha_1{}^{(k+a_1-1)}e^{-\alpha_1(T_{2k}+b_1)}\alpha_2{}^{(l-k+a_2-1)}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}}{\sum_k\alpha_1{}^{(k+a_1-1)}e^{-\alpha_1(T_{2k}+b_1)}\alpha_2{}^{(l-k+a_2-1)}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}}$$

$$\rho(k|\underline{x}) = \frac{\int_0^\infty e^{-\alpha_1(T_{2k}+b_1)}\alpha_1{}^{(k+a_1-1)}\,d\alpha_1\int_0^\infty e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}\alpha_2{}^{(l-k+a_2-1)}\,d\alpha_2}{\sum_k\int_0^\infty e^{-\alpha_1(T_{2k}+b_1)}\alpha_1{}^{(k+a_1-1)}\,d\alpha_1\int_0^\infty e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}\alpha_2{}^{(l-k+a_2-1)}\,d\alpha_2}$$

Assuming $\alpha_1(T_{2\alpha}+b_1) = y\quad \&\alpha_2(T_{2l}-T_{2k}+b_2) = z$

$$\alpha_1 = \frac{y}{(T_{2k}+b_1)}\&\alpha_2 = \frac{z}{T_{2l}-T_{2k}+b_2}$$

$$d\alpha_1 = \frac{dy}{(T_{2k}+b_1)}\&d\alpha_2 = \frac{z}{T_{2l}-T_{2k}+b_2}$$

$$\rho(k|\underline{x}) = \frac{\int_0^\infty e^{-y}\frac{y^{(k+a_1-1)}}{(T_{2k}+b_1)^{(k+a_1-1)}}\frac{dy}{(T_{2k}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{z}}\frac{\mathrm{z}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2-1)}}\frac{dz}{(T_{2l}-T_{2k}+b_2)}}{\sum_k\int_0^\infty e^{-y}\frac{y^{(k+a_1-1)}}{(T_{2k}+b_1)^{(k+a_1-1)}}\frac{dy}{(T_{2k}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{z}}\frac{\mathrm{z}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2-1)}}\frac{dz}{(T_{2l}-T_{2k}+b_2)}}$$

$$\rho(k|\underline{x}) = \frac{\frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}{\sum_k\frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}$$

$$\rho(k|\underline{x}) = \frac{\frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}{\xi(a_{1,}a_{2,}b_{1,}b_{2,}k,l)} \quad (1.6.1.13)$$

The marginal posterior distribution of α_1, using equations(1.6.1.6) and (1.6.1.7)

$$\rho(\alpha_1|\underline{x}) = \frac{\mathrm{L}(\alpha_{1,}\alpha_2/\underline{\mathrm{x}})\,\pi(\alpha_1)}{\int_0^\infty \mathrm{L}(\alpha_{1,}\alpha_2/\underline{\mathrm{x}})\,\pi(\alpha_1)\mathrm{d}\alpha_1}$$

$$\rho(\alpha_1|\underline{x}) = \frac{\sum_k\int_0^\infty \mathrm{L}(\alpha_{1,}\alpha_2/\underline{\mathrm{x}})\,\pi(\alpha_1)\,\pi(\alpha_2)\,\mathrm{d}\alpha_2}{\sum_k\iint_0^\infty \mathrm{L}(\alpha_{1,}\alpha_2/\underline{\mathrm{x}})\,\pi(\alpha_1)\,\pi(\alpha_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}$$

On solving which gives

$$\rho(\alpha_1|\underline{x}) = \frac{\sum_k e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)} \int_0^\infty e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\alpha_2}^{(l-k+a_2-1)}\, d\alpha_2}{\sum_k \int_0^\infty e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)}\, d\alpha_1 \int_0^\infty e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\alpha_2}^{(l-k+a_2-1)}\, d\alpha_2}$$

Assuming $\alpha_1(T_{2k}+b_1) = y$ $\quad \& \alpha_2(T_{2l}-T_{2k}+b_2) = z$

$$\alpha_1 = \frac{y}{(T_{2k}+b_1)} \& \alpha_2 = \frac{z}{T_{2l}-T_{2k}+b_2}$$

$$d\alpha_1 = \frac{dy}{(T_{2k}+b_1)} \& \mathrm{d}\alpha_2 = \frac{dz}{T_{2l}-T_{2k}+b_2}$$

$$\rho(\alpha_1|\underline{x}) = \frac{\sum_k e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)} \int_0^\infty \mathrm{e}^{-\mathrm{z}} \frac{\mathrm{z}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2-1)}} \frac{dz}{(T_{2l}-T_{2k}+b_2)}}{\sum_k \int_0^\infty e^{-y} \frac{y^{(k+a_1-1)}}{(T_{2k}+b_1)^{(k+a_1-1)}} \frac{dy}{(T_{2k}+b_1)} \int_0^\infty \mathrm{e}^{-\mathrm{z}} \frac{\mathrm{z}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2-1)}} \frac{dz}{(T_{2l}-T_{2k}+b_2)}}$$

$$\rho(\alpha_1|\underline{x}) = \frac{\sum_m e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)} \frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}$$

$$\rho(\alpha_1|\underline{x}) = \frac{\sum_k e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)} \frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}{\xi(a_{1,}a_{2,}b_{1,}b_{2,}k,l)} \quad (1.6.1.14)$$

The marginal posterior distribution of α_2, using the equation (1.6.1.6)&(1.6.1.8) is

$$\rho(\alpha_2|\underline{x}) = \frac{\mathrm{L}(\alpha_{1\,,}\alpha_2/\underline{\mathrm{x}})\, \pi(\alpha_2)}{\int_0^\infty \mathrm{L}(\alpha_{1\,,}\alpha_2/\underline{\mathrm{x}})\, \pi(\alpha_2)\, \mathrm{d}\alpha_2}$$

$$\rho(\alpha_2|\underline{x}) = \frac{\sum_k \int_0^\infty \mathrm{L}(\alpha_{1\,,}\alpha_2/\underline{\mathrm{x}})\, \pi(\alpha_1)\, \pi(\alpha_2)\, \mathrm{d}\alpha_1}{\sum_k \iint_0^\infty \mathrm{L}(\alpha_{1\,,}\alpha_2/\underline{\mathrm{x}})\, \pi(\alpha_1)\, \pi(\alpha_2)\, \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2}$$

$$\rho(\alpha_2|\underline{x}) = \frac{\sum_k e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\alpha_2}^{(l-k+a_2-1)} \int_0^\infty e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)}\, d\alpha_1}{\sum_k \int_0^\infty e^{-\alpha_1(T_{2k}+b_1)}{\alpha_1}^{(k+a_1-1)}\, d\alpha_1 \int_0^\infty e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\theta_2}^{(l-k+a_2-1)}\, d\alpha_2}$$

Assuming $\alpha_1(T_{2k}+b_1) = y$ $\quad \& \alpha_1 = \frac{y}{(T_{2k}+b_1)}$

$$\rho(\alpha_2|\underline{x}) = \frac{\sum_k e^{-\theta_2(T_{2l}-T_{2k}+b_2)}{\theta_2}^{(l-k+a_2-1)} \int_0^\infty e^{-y} \frac{y^{(k+a_1-1)}}{(T_{2k}+b_1)^{(k+a_1-1)}} \frac{dy}{(T_{2k}+b_1)}}{\sum_k \int_0^\infty e^{-y} \frac{y^{(k+a_1-1)}}{(T_{2k}+b_1)^{(k+a_1-1)}} \frac{dy}{(T_{2k}+b_1)} \int_0^\infty \mathrm{e}^{-\mathrm{z}} \frac{\mathrm{z}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2-1)}} \frac{dz}{(T_{2l}-T_{2k}+b_2)}}$$

$$\rho(\alpha_2|\underline{x}) = \frac{\sum_k \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}} e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\alpha_2}^{(l-k+a_2-1)}}{\sum_k \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}} \frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}}$$

$$\rho(\alpha_2|\underline{x}) = \frac{\sum_k \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}} e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\alpha_2}^{(l-k+a_2-1)}}{\xi(a_{1,}a_{2,}b_{1,}b_{2,}k,l)} \quad (1.6.1.15)$$

1.6.2 Bayes Estimators under Linex Loss Function (LLF)

The Bayes estimate $\hat{k}_{BL}$ of m under LLF using marginal posterior of equation (1.6.1.12), is given as

$$\hat{k}_{BL} = -\frac{1}{k_1} \log \left[\frac{\sum_k e^{-k_1 k} \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}} e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}{\alpha_2}^{(l-k+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,k,l)} \right] \quad (1.6.2.1)$$

The Bayes estimate of $\hat{k}_{1BL}$ of k_1 using marginal posterior of equation (1.6.1.13) under LLF equation (1.6.2.1) is given by

$$\hat{\alpha}_{1BL} = -\frac{1}{k_1} \log E_\rho[\exp(-k_1\alpha_1)]$$

$$\hat{\alpha}_{1BL} = -\frac{1}{k_1} \log \left[\frac{\sum_k \frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}} \int_0^\infty e^{-\alpha_1(T_{2k}+b_1+k_1)}{\alpha_1}^{(k+a_1-1)} d\alpha_1}{\xi(a_1, a_2, b_1, b_2, k.l)} \right]$$

Assuming $\alpha_1(T_{2k}+b_1+k_1) = y$ $\quad \& \alpha_1 = \frac{y}{(T_{2m}+b_1+k_1)}$

$$\hat{\alpha}_{1BL} = -\frac{1}{k_1} \log \left[\frac{\sum_k \frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}} \int_0^\infty e^{-y} \frac{y^{(k+a_1-1)}}{(T_{2k}+b_1+k_1)^{(k+a_1-1)}} \frac{dy}{(T_{2k}+b_1+k_1)}}{\xi(a_1, a_2, b_1, b_2, k.l)} \right]$$

$$\hat{\alpha}_{1BL} = -\frac{1}{k_1}\log\left[\frac{\sum_k \frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2)^{(l-k+a_2)}}\frac{\Gamma(k+a_1)}{(T_{2k}+b_1+k_1)^{(k+a_1)}}}{\xi(a_1,a_2,b_1,b_2,k,l)}\right]$$

$$\hat{\alpha}_{1BL} = -\frac{1}{k_1}\log\left[\frac{\xi[a_1,a_2,(b_1+k_1),b_2,k,l]}{\xi(a_1,a_2,b_1,b_2,k,l)}\right] \quad (1.6.2.2)$$

The Bayes estimate of $\hat{\alpha}_{2BL}$of α_2 using marginal posterior of equation (1.6.1.16) under LLF equation (1.6.2.1) is given by

$$\hat{\alpha}_{2BL} = -\frac{1}{k_2}\log E_\rho[\exp(-k_2\alpha_2)]$$

$$\hat{\alpha}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\sum_m e^{-k_2\alpha_2}\frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}e^{-\alpha_2(T_{2l}-T_{2k}+b_2)}\theta_2{}^{(l-k+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,k,l)}\right]$$

$$\hat{\alpha}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\sum_k \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\int_0^\infty e^{-\alpha_2(T_{2l}-T_{2k}+b_2+k_2)}\theta_2{}^{(k-l+a_2-1)}d\theta_2}{\xi(a_1,a_2,b_1,b_2,k,l)}\right]$$

Assuming $\alpha_2(T_{2l}-T_{2k}+b_2+k_2) = y$ &$\alpha_2 = \frac{y}{(T_{2l}-T_{2k}+b_2+k_2)}$

Then

$$\hat{\alpha}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\sum_k \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\int_0^\infty \mathrm{e}^{-\mathrm{y}}\frac{\mathrm{y}^{(l-k+a_2-1)}}{(T_{2l}-T_{2k}+b_2+k_2)^{(l-k+a_2-1)}}\frac{dy}{(T_{2l}-T_{2k}+b_2)}}{\xi(a_1,a_2,b_1,b_2,k,l)}\right]$$

$$\hat{\alpha}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\sum_k \frac{\Gamma(k+a_1)}{(T_{2k}+b_1)^{(k+a_1)}}\frac{\Gamma(l-k+a_2)}{(T_{2l}-T_{2k}+b_2+k_2)^{(l-k+a_2)}}}{\xi(a_1,a_2,b_1,b_2,k,l)}\right]$$

$$\hat{\alpha}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\xi[a_1,a_2,b_1,(b_2+k_2),k,l]}{\xi(a_1,a_2,b_1,b_2,k,l)}\right] \quad (1.6.2.4)$$

Numerical Comparison for Exponentiated Inverted Weibull Sequences

In this paper we have generated 20 random observations from Exponentiated Inverted Weibull distribution with scale parameter α = 2. and β = 0.5. The observed data mean is μ = 1.5616 and variance σ^2 = 0.6812. Let the change in sequence is at 11^{th} observation, so the means and variances of both sequences ($x_1, x_2, \ldots, x_k$) and ($x_{(k+1)}, x_{(k+2)}, \ldots, x_l$) are μ_1= 1.5491, μ_2= 1.5769, σ_1^2 = 1.0197 and σ_2^2= 0.3427. If the target value of μ_1 is unknown, its estimating ($\hat{\mu}_1$) is given by the mean of first k sample observation given k=13, μ = 1.5491.

Sensitivity Analysis of Bayes Estimates

In this section we have studied the sensitivity of the Bayes estimates with respect to changes in the parameters of prior distribution a_1, b_1, a_2 and b_2. The means and variances of the prior distribution are used as prior information in computing these parameters. Then with these parameter values we have computed the Bayes estimates of k, α_1 and α_2 under LLF considering different set of values of $(a_1, b_1)$and (a_2, b_2).We have also considered the other values like parameter of loss function α_1=2 and different sample sizes n=10(10)30. The Bayes estimates of the change point 'k' and the parameters α_1 and α_2 are given in table-1 under LLF.. Their respective mean squared errors (M.S.E's) are calculated by repeating this process 1000 times and presented in same table in small parenthesis under the estimated values of parameters. All these values appears to be robust with respect to correct choice of prior parameter values and appropriate sample size. All the estimators perform better with sample size n=20.Similarly the Bayes estimates of LLF are presented in table 1 appears to be sensitive with wrong choice of prior parameters and sample size. All the calculations are done by R-programming. From the below two table we conclude that–

The Bayes estimates of the parameters α_1 and α_2 of EIW obtained with loss function LLF have more or less same numerical values. The respective M.S.E's shows that the Bayes estimates uniformly smaller for $\hat{\alpha}_{1BL}$ and $\hat{\alpha}_{2BL}$ under LLF except of $\hat{k}_{BL}$. The Bayes estimates of the parameters are robust uniformly for all values of prior parameters and all sample size.

Table 1.1
Bayes Estimates of m,α_1&$\alpha_2$for EIW sequences and their respective M.S.E.'s Under LLF

(a_1, b_1)	(a_2, b_2)	n	$\hat{k}_{BL}$	$\hat{\alpha}_{1BL}$	$\hat{\alpha}_{2BL}$
(1.25,1.50)	(1.50,1.60)	10	7.8567 **(18.5629)**	2.4152 **(0.0423)**	5.6399 **(0.0533)**
		20	17.4757 **(18.8058)**	3.6054 **(0.0723)**	2.8843 **(0.1317)**
		30	27.2234 **(18.5613)**	1.8391 **(0.1517)**	2.3582 **(0.0604)**
(1.50,1.75)	(1.70,1.80)	10	8.0800 **(20.3466)**	2.8315 **(0.0675)**	1.9187 **(0.0699)**
		20	17.2980 **(17.0093)**	1.8214 **(0.2639)**	3.2813 **(0.0396)**
		30	27.0351 **(17.9915)**	1.3656 **(0.0655)**	1.8160 **(0.0762)**
(1.75,2.0)	(1.90,2.0)	10	7.9206 **(19.0464)**	3.1722 **(0.0917)**	1.6689 **(0.0473)**
		20	17.5442 **(18.6643)**	2.0446 **(0.2046)**	2.2066 **(0.5266)**
		30	27.5183 **(18.0671)**	2.9711 **(0.1095)**	1.2995 **(0.0156)**
(2.0,2.25)	(2.10,2.20)	10	7.7801 **(18.4644)**	2.2183 **(0.0370)**	2.0520 **(0.0542)**
		20	17.7479 **(18.1044)**	2.7108 **(0.0451)**	2.8612 **(0.1935)**
		30	27.5316 **(20.1331)**	2.6096 **(0.0052)**	1.9407 **(0.3302)**
(2.25,2.50)	(2.30,2.40)	10	8.1371 **(18.0578)**	2.4303 **(0.0003)**	1.5661 **(0.1830)**
		20	17.7958 **(123.4386)**	2.1493 **(0.0910)**	2.4853 **(0.0192)**
		30	27.6648 **(328.5269)**	1.8557 **(0.0049)**	1.5343 **(0.0053)**
(2.50,2.75)	(2.50,2.60)	10	7.9146 **(17.3859)**	1.7218 **(0.2007)**	2.1832 **(0.0009)**
		20	18.1318 **(124.6488)**	2.0797 **(0.0874)**	1.8301 **(0.1783)**
		30	27.7977 **(322.3771)**	2.5156 **(0.1225)**	1.9650 **(0.0043)**

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