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Fekete Szego Bounds for a Class of Regular P-Valent Functionsthat Is Subclass of Class of P-Valent Convex Functions

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ABSTRACT: *Here we describe some classes of analytic functions and its subclasses by which we will be obtaining sharp upper bounds of the functional* $|a_3 - \square a_2^2|$ *for the analytic function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ *belonging to these classes and subclasses.*
KEYWORDS: *Univalent functions, Starlike functions, Close to convex functions and bounded functions.*
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I. Introduction :

Let $\boldsymbol{\mathcal{A}}$ denote the class of functions of the form
$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)
which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\boldsymbol{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$. In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \boldsymbol{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \boldsymbol{S}$..

With the known estimates $|a_2| \leq 2$and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\boldsymbol{S}$,Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\boldsymbol{S}$.

Let $f(z) \in \boldsymbol{S}$, then

$$|a_3 - \square a_2^2| \leq \begin{cases} 3 - 4\square, if\ \square \leq 0; \\ 1 + 2\exp\left(\dfrac{-2\square}{1-\square}\right), if\ 0 \leq \square \leq 1; \\ 4\square - 3, if \square \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\boldsymbol{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\boldsymbol{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \boldsymbol{\mathcal{A}}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by $\mathfrak{C}$ and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{ f(z) \in \boldsymbol{\mathcal{A}}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \tag{1.6}$$

$$\mathcal{K}(A,B) = \left\{ f(z) \in \boldsymbol{\mathcal{A}}; \frac{\left(zf'(z)\right)'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \tag{1.7}$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as

$$\left\{ f(z) \in \mathcal{A}; \frac{z(zf'(z))'}{z(f'(z))} \prec\ p\frac{1+w(z)}{1-w(z)}; z \in \mathbb{E} \right\}$$

and we will denote this class as $f(z) \in KS_p^*]$,

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z));\ z \in \mathbb{E}$ and we write $f(z)\ \prec\ F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. \tag{1.8}$$

It is known that

$$|d_1| \leq 1, |d_2| \leq\ 1 - |d_1|^2. \tag{1.9}$$

II. PRELIMINARY LEMMAS:

For $0\ <\ c\ <\ 1$, we write $w(z)\ =\ \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)}\ =\ 1\ +\ 2c_1 z +\ 2(c_2 + c_1^2)z^2 + - - - (2.1)$$

III. MAIN RESULTS

<u>THEOREM 3.1</u>: Let $f(z) \in KS_p^*$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \left[\dfrac{p^2(2p+1)}{p+2} - \dfrac{\mu 4p^4}{(p+1)^2}\right]; \ if\ \ \mu \leq\ \dfrac{(p+1)^2}{2p(p+2)} & (3.1) \\ \dfrac{p^2}{(p+2)}\ ;\ \ if\ \ \dfrac{(p+1)^2}{2p(p+2)} \leq \mu \leq\ \dfrac{(p+1)^3}{2p^2(p+2)} & (3.2) \\ \left[\dfrac{4p^4\mu}{(p+1)^2} - \dfrac{p^2(2p+1)}{p+2}\right]; \ if\ \ \mu \geq\ \dfrac{(p+1)^3}{2p^2(p+2)} & (3.3) \end{cases}$$

The results are sharp.

<u>Proof</u>: By definition of $f(z) \in KS_p^*$, , we have

$$\frac{\mathrm{z(zf'(z))'}}{\mathrm{z(f'(z))}} = \mathrm{p}\frac{1+\mathrm{w(z)}}{1-\mathrm{w(z)}}\ \ ;\ \ \mathrm{w(z)} \in \mathcal{U}. \tag{3.4}$$

Expanding the series (3.4), we get

$$\left\{p^2 z^p + (p+1)^2 a_{p+1}\, z^{p+1} + (p+2)^2 a_{p+2}\, z^{p+2} - - \right\} =\ p\ (1 +\ 2c_1 z +\ 2(c_2 + c_1^2)z^2 + - -)(3.5)$$

Identifying terms in (3.5), we get

$$a_{p+1} = \frac{2p^2}{(p+1)} c_1 \tag{3.6}$$

$$a_{p+1} = \frac{p^2}{(p+2)} c_2 + \frac{[p^2(2p+1))]}{(p+2)} {c_1}^2. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(p)^2}{(p+2)} c_2 + \left[\frac{2p^3+p^2}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right] c_1^2 \tag{3.8}$$

Taking absolute value, (3.8) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(p)^2}{(p+2)} |c_2| + \left|\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right| |c_1^2|. \tag{3.9}$$

Using (1.11) in (3.9), we get

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(p)^2}{(p+2)} (1 - |c_1|^2) + \left|\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right| |c_1^2|$$

$$= \frac{(p)^2}{(p+2)} + \left[\left|\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right| - \frac{(p)^2}{(p+2)}\right] |c_1|^2. \tag{3.10}$$

<u>Case I</u>:$\mu \leq \frac{(p+1)^2(2p+1)}{4p^2(p+2)}$.

(3.10) can be rewritten as

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} + \left[\frac{2(p)^3}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right] |c_1|^2. \tag{3.11}$$

<u>Subcase I (a)</u>: $\mu \leq \frac{(p+1)^2}{2p(p+2)}$. Using (1.11), (3.11) becomes

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \left[\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right] \tag{3.12}$$

<u>Subcase I (b)</u>: $\mu \geq \frac{(p+1)^2}{2p(p+2)}$. We obtain from (3.11)

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)}. \tag{3.13}$$

<u>Case II</u>: $\mu \geq \frac{(p+1)^2(2p+1)}{4p^2(p+2)}$

Preceding as in case I, we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} + \left[\frac{4p^4\mu}{(p+1)^2} - \frac{p^2(p+1)}{(p+2)}\right] |c_1|^2. \tag{3.14}$$

<u>Subcase II (a)</u>: $\mu \leq \frac{(p+1)^3}{2p^2(p+2)}$

(3.14) takes the form

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} \tag{3.15}$$

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} \;\; ; \;\; if \;\; \frac{p^2(2p+1)}{(p+2)} \leq \mu \leq \;\; \frac{(p+1)^3}{2p^2(p+2)} \tag{3.16}$$

<u>**Subcase II (b)**</u>: $\mu \;\geq \frac{(p+1)^3}{2p^2(p+2)}$

Preceding as in subcase I (a), we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \left[\frac{4p^4\mu}{(p+1)^2} - \frac{p^2(2p+1)}{(p+2)}\right] \tag{3.17}$$

<u>Combining (3.12), (3.16) and (3.17), the theorem is proved</u>.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = \frac{z^p}{p} + \frac{p\, z^{p+1}}{p+1} + \frac{p(p-1)}{2!} \;\; \frac{z^{p+2}}{p+2} + ------$$

Extremal function for (3.2) is defined by

$$f_2(z) = \frac{z^p}{p} + \frac{p\, z^{p+2}}{p+2} + \frac{p(p-1)}{2!} \;\; \frac{z^{p+4}}{p+4} + -----.$$

Corollary 3.2: Putting $p = 1$ in the theorem, we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} 1-\mu, if \mu \leq \dfrac{2}{3}; \\ \dfrac{1}{3} if \dfrac{2}{3} \leq \mu \leq \dfrac{4}{3}; \\ \mu - 1, if \; \mu \geq \dfrac{4}{3} \end{cases}$$

These are the required results of class KS*.

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