International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 10 Issue 10 ǁ October 2022 ǁ PP. 504-508

Coefficient Inequality for a Newly Constructed Subclass of Class of Starlike Analytic Functions

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ABSTRACT: *In this paper, we will discuss a newly constructed subclass of analytic starlikefunctions by which we will be obtaining sharp upper bounds of the functional* $|a_3 - \square a_2^2|$ *for the analytic function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ *belonging to this subclasses.*

KEYWORDS: *Univalent functions, Starlike functions, Close to convex functions and bounded functions.*
MATHEMATICS SUBJECT CLASSIFICATION: 30C50

Date of Submission: 08-10-2022 Date of acceptance: 18-10-2022

I. Introduction :

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$,Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \square a_2^2| \leq \begin{cases} 3 - 4\square, if\ \square \leq 0; \\ 1 + 2e^{\frac{-2\square}{1-\square}}, if\ 0 \leq \square \leq 1; \\ 4\square - 3, if \square \geq 1. \end{cases} (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$

and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$

and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B)=\left\{f(z)\in\boldsymbol{\mathcal{A}};\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,z\in\mathbb{E}\right\}\qquad(1.6)$$

$$\mathcal{K}(A,B)=\left\{f(z)\in\boldsymbol{\mathcal{A}};\frac{\left(zf'(z)\right)'}{f'(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,z\in\mathbb{E}\right\}\qquad(1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.
We introduce a new subclassas

$$\left\{f(z)\in\mathcal{A};\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)\prec\frac{1+z}{1-z};z\in\mathbb{E}\right\}$$

and we will denote this class as $f(z)\in\Sigma S^*[\alpha]$.
Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z));\ z\in\mathbb{E}$ and we write $f(z)\prec F(z)$.
By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z)=\sum_{n=1}^{\infty}d_nz^n,w(0)=0,|w(z)|<1(1.8)$$

It is known that

$$|d_1|\le 1,|d_2|\le 1-|d_1|^2(1.9)$$

II. PRELIMINARY LEMMAS:

For $0<c<1$, we write

$$w(z)=\left(\frac{c+z}{1+cz}\right)$$

so that

$$\frac{1+w(z)}{1-w(z)}=1+2c_1z+2(c_2+c_1^2)z^2+---\qquad(2.1)$$

III. MAIN RESULTS

THEOREM 3.1: Let $f(z)\in f(z)\in\Sigma S^*[\alpha]$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{2\alpha}{(\alpha+1)^3}[5\alpha^2+10\alpha-3-8\square\alpha(\alpha+1)];if\ \square\le\dfrac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}(3.1)\\[2ex]\dfrac{2\alpha}{\alpha+1}\quad;\ if\ \dfrac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}\le\square\le\dfrac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}(3.2)\\[2ex]\dfrac{2\alpha}{(\alpha+1)^3}[8\square\alpha(\alpha+1)-5\alpha^2-10\alpha+3]\ ;if\ \square\ge\dfrac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}(3.3)\end{cases}$$

The results are sharp.
Proof: By definition of$f(z)\in f(z)\in\Sigma S^*[\alpha]$, we have

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)=\frac{1+w(z)}{1-w(z)};w(z)\in\mathcal{U}.\qquad(3.4)$$

Expanding the series (3.4), we get

$$1+a_2z\left(\frac{\alpha+1}{2\alpha}\right)+\frac{z^2}{2}\left[(2a_3-a_2{}^2)\left(\frac{\alpha+1}{\alpha}\right)+\left(\frac{1-\alpha}{2\alpha^2}\right)a_2^2\right]+---$$
$$=(1+2c_1z+2(c_1^2+c_2)z^2+\mathrm{z}^3(2c_3+4c_1c_{2+}c_1{}^3)+---).\quad(3.5)$$

Identifying terms in (3.5), we get

$$a_2=\frac{4\alpha c_1}{\alpha+1}\qquad(3.6)$$

$$a_3=\left(\frac{2\alpha}{\alpha+1}\right)\left[c_1^2+c_2+\frac{4c_1^2}{(\alpha+1)^2}[\alpha^2+2\alpha-1]\right]\qquad(3.7)$$

From (3.6) and (3.7), we obtain

$$a_3-\mu a_2^2=c_1^2\left[\frac{2\alpha}{\alpha+1}+\frac{8\alpha(\alpha^2+2\alpha-1)}{(\alpha+1)^3}-\frac{16\mu\alpha^2}{(\alpha+1)^2}\right]+c_2\left[\frac{2\alpha}{\alpha+1}\right]\qquad(3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \left|\frac{2\alpha}{\alpha+1} + \frac{8\alpha(\alpha^2+2\alpha-1)}{(\alpha+1)^3} - \frac{16\mu\alpha^2}{(\alpha+1)^2}\right| |c_1^2| + |c_2| \left|\frac{2\alpha}{\alpha+1}\right| \qquad (3.9)$$

Using (1.11) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \left[\frac{2\alpha}{(\alpha+1)^3}|(5\alpha^2+10\alpha-3) - 8\mu\alpha(\alpha+1)| - \frac{2\alpha}{\alpha+1}\right]|c_1|^2 \; (3.10)$$

Case I:$\mu \geq \frac{5\alpha^2+10\alpha-3}{8\alpha(\alpha+1)}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \frac{2\alpha}{(\alpha+1)^3}[8\mu\alpha(\alpha+1) - (6\alpha^2+12\alpha-2)]|c_1|^2 (3.11)$$

Subcase I (a): $\mu \geq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}$. Using (1.11), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{(\alpha+1)^3}[8\mu\alpha(\alpha+1) - 5\alpha^2 - 10\alpha + 3]. \qquad (3.12)$$

Subcase I (b): $\mu \leq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} \qquad (3.13)$$

Case II: $\mu \leq \frac{5\alpha^2+10\alpha-3}{8\alpha(\alpha+1)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \frac{2\alpha}{(\alpha+1)^3}[4\alpha^2+8\alpha-4-8\mu\alpha(\alpha+1)]|c_1|^2. \quad (3.14)$$

Subcase II (a): $\mu \leq \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}$

(3.14) takes the form(3.15)

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{(\alpha+1)^3}[5\alpha^2+10\alpha-3-8\mu\alpha(\alpha+1)] \qquad (3.15)$$

Subcase II (b): $\mu \geq \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = z\left(1 + \frac{59\alpha^3 - 15\alpha^2 - 7\alpha + 3}{4\alpha(\alpha+1)^2}z\right)^{\frac{64\alpha^3}{59\alpha^3-15\alpha^2-7\alpha+3}}$$

Extremal function for (3.2) is defined by $f_2(z) = z(1+z^2)^{\frac{2\alpha}{\alpha+1}}$.

Corollary 3.2: Putting $\alpha = 1$, in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if \mu \leq \frac{1}{2}; \\ 1 if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if \; \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

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