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Multidimensional of gradually series band limited functions and frames

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ABSTRACT: We show frames for $L_2[-\pi,\pi]^d$ consisting of exponential functions in connection to over sampling and nonuniform sampling of gradually series bandlimited functions. We derive a multidimensional nonuniform over sampling formula for gradually series bandlimited functions with a fairly general frequency domain. The stability of this formula under various perturbations in the sampled data is investigated, and a computationally manageable simplification of the main over sampling theorem is given. Also, a generalization of Kadec's 1/4 theorem to higher dimensions is considered. Finally, the developed techniques are used to approximate biorthogonal functions of particular exponential Riesz bases for $L_2[-\pi,\pi]$,and a well known theorem of Levinson is recovered as a corollary.

Keywords: Frames Sampling Kadec Levinson Riesz basis Over sampling

I. INTRODUCTION

My methods and uses follows the methodology of [14]. The recovery of gradually series bandlimited signals from discrete data has its origins in the Whittaker– Kotel'nikov –Shannon (WKS) sampling theorem the first and simplest such recovery formula. The formula recovers a function with a frequency band of $[-\pi,\pi]$ given the function's values at the integers. The WKS theorem has drawbacks. Foremost, the recovery formula does not converge given certain types of error in the sampled data, as Daubechies and DeVore mention in [7]. They use over sampling to derive an alternative recovery formula which does not have this defect. Additionally for the WKS theorem, the data nodes have to be equally spaced, and nonuniform sampling nodes are not allowed.Their sampling formulae recover a function from nodes $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)}$, where $\left(e^{it_{(m+\epsilon_0)}x}\right)_{(m+\epsilon_0)}, \epsilon_0 > 0$ forms a Riesz basis for $L_2[-\pi,\pi]$. More generally, frames have been applied to nonuniform sampling, particularly in the work of Benedetto and Heller in [2,3]; see also.

Benjamin Bailey [13] derive a multidimensional over sampling formula (see (4)), for nonuniform nodes and bandlimited functions with a fairly general frequency domain; we investigate the stability of (4) under perturbation of the sampled data. We present a computationally feasible version of (4) in the case where the nodes are asymptotically uniformly distributed. Kadec's theorem gives a criterion for the nodes $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ so that $\left(e^{it_{(m+\epsilon_0)}x}\right)_{(m+\epsilon_0)}$ forms a Riesz basis for $L_2[-\pi,\pi]$.Generalizations of Kadec's 1/4 theorem to higher dimensions are considered [13], and an asymptotic equivalence of two generalizations is given. We investigate an approximation of the biorthogonal functionals of Riesz bases. Additionally, we give a simple proof of a theorem of Levinson.

II. PRELIMINARIES

We use the d-dimensional L_2 Fourier transform

$$\mathcal{F}(f)(\cdot) = \int_{\mathbb{C}^d} f(\xi) e^{-i\langle\cdot,\xi\rangle}\, d\xi, f \in L_2(\mathbb{C})^d,$$

where the inverse transform is given by

$$\mathcal{F}^{-1}(f)(\cdot) = \frac{1}{(2\pi)^d}\int_{\mathbb{C}^d} f(\xi) e^{i\langle\cdot,\xi\rangle}\, d\xi, f \in L_2(\mathbb{C})^d,$$

the integral is actually a principal value where the limit is in the L_2 sense. This map is an onto isomorphism from $L_2(\mathbb{C}^d)$ to itself.

Definition 2.1 . Given a bounded measurable set E with positive measure, we define

$$PW_E = \{f_j \in L_2(\mathbb{C}^d) \,|\, supp(\mathcal{F}^{-1}(f_j)) \subset E\}.$$

Functions in PW_E are said to be gradually series bandlimited.

Definition 2.2 . The function sinc : $\mathbb{C} \to \mathbb{C}$ is defined by $sinc(x) = \frac{sin(x)}{x}$. We also define the multidimensional sinc function

SINC : $\mathbb{C}^d \to \mathbb{C}^d$ by $\mathrm{SINC}(x) = \mathrm{sinc}(x_1) \cdots \mathrm{sinc}(x_d), x = (x_1, \ldots, x_d)$.
We recall some basic facts about PW_E :
(i) PW_E is a Hilbert space consisting of entire functions, though in this paper we only regard the functions as having real arguments.
(ii) In PW_E , L_2 convergence implies uniform convergence. This is an easy consequence of the Cauchy–Schwarz inequality.
(iii) The function $sinc(\pi(x - y))$ is a reproducing kernel for $PW_{[-\pi,\pi]}$ That is, if $\in PW_{[-\pi,\pi]}$, then we have

$$\sum_j f_j(t) = \int_{-\infty}^{\infty} \sum_j f_j(\tau) sinc\pi(t-\tau) d\tau , t \in \mathbb{C}. \quad (1)$$

(iv) The WKS sampling theorem (see for example [69, p. 91]): If $f \in PW_{[-\pi,\pi]}$,then

$$\sum_j f_j(t) = \sum_{(m+\epsilon_0)\in\mathbb{Z}} \sum_j f_j((m+\epsilon_0)) \, sinc\pi(t-(m+\epsilon_0)) , t \in \mathbb{C},$$

where the sum converges in $PW_{[-\pi,\pi]}$, and hence uniformly.

If $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$ is a Schauder basis for a Hilbert space H, then there exists a unique set of functions $(f^*_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$ (the biorthogonalsof $\left((f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}\right)$such that $\langle f_{(m+\epsilon_0)}, f^*_m \rangle = \delta_{(m+\epsilon_0)m}$. The biorthogonals also form a Schauder basis for H. Note that biorthogonality is preserved under a unitary transformation.

Definition 2.3 . A sequence $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)} \subset H$ such that the map $Le_{(m+\epsilon_0)} = f_{(m+\epsilon_0)}$ is an onto isomorphism is called a Riesz basis for H. The following definitions and facts concerning frames are found in [6, Section 4].

Definition . 4 . A gradually series frame for a separable Hilbert space H is a sequence $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)} \subset H$ such that for some $A > 0$,

$$A \sum_j \|f_j\|^2 \leq \sum_{(m+\epsilon_0)} \sum_j |\langle f_j, f_{(m+\epsilon_0)} \rangle|^2 \leq (A+\epsilon_1) \sum_j \|f_j\|^2, \qquad \forall f_j \in H, \epsilon_1 > 0. \quad (2)$$

The numbers A and $(A+\epsilon_1)$ in (2) are called the lower and upper frame bounds. Let H be a Hilbert space with orthonormal basis $(e_{(m+\epsilon_0)})_{(m+\epsilon_0)}$. The following conditions are equivalent to $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)} \subset H$ being a gradually series frame for H.
(i) The map $L : H \to H$ defined by $Le_{(m+\epsilon_0)} = f_{(m+\epsilon_0)}$ is bounded linear and onto. This map is called the synthesis operator.
(ii) The map $L^* : H \to H$ (the analysis operator) given by $f_j \mapsto \sum_{(m+\epsilon_0)} \langle f_j, f_{(m+\epsilon_0)} \rangle e_{(m+\epsilon_0)}$ is an isomorphic embedding.
Given a gradually series frame $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ with synthesis operator L, the map $S = LL^*$ given by $S \sum_j f_j = \sum_{(m+\epsilon_0)} \sum_j \langle f_j, f_{(m+\epsilon_0)} \rangle f_{(m+\epsilon_0)}$ is an onto isomorphism. S is called the frame operator associated to the frame. It follows that S is positive and self-adjoint. The basic connection between frames and sampling theory of gradually series bandlimited functions (more generally in a reproducing kernel Hilbert space) is straightforward. If $(e^{it_{(m+\epsilon_0)}(\cdot)})_{(m+\epsilon_0)}$ is a frame for $f_j \in PW_{[-\pi,\pi]}$ with frame operator S, and $f_j \in PW_{[-\pi,\pi]}$, then

$$S \sum_j \left(\mathcal{F}^{-1}(f_j)\right) = \sum_{(m+\epsilon_0)} \sum_j \langle \mathcal{F}^{-1}(f_j), f_{(m+\epsilon_0)} \rangle f_{(m+\epsilon_0)} = \sum_{(m+\epsilon_0)} \sum_j \mathcal{F}\left(\mathcal{F}^{-1}(f_j)\right)(t_{(m+\epsilon_0)}) f_{(m+\epsilon_0)}$$
$$= \sum_{(m+\epsilon_0)} \sum_j f_j(t_{(m+\epsilon_0)}) f_{(m+\epsilon_0)} ,$$

implying that $\sum_j \mathcal{F}^{-1}(f_j) = \sum_{(m+\epsilon_0)} \sum_j f_j(t_{(m+\epsilon_0)}) S^{-1} f_{(m+\epsilon_0)}$, so that $\sum_j f_j = \sum_{(m+\epsilon_0)} \sum_j f_j(t_{(m+\epsilon_0)}) \mathcal{F}^{-1}(S^{-1} f_{(m+\epsilon_0)})$. Note that in the case when $t_{(m+\epsilon_0)} = (m+\epsilon_0)$, we recover the WKS theorem .

Definition 2. 5 .A sequence $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ satisfying the second inequality in (37) is called a Bessel sequence.

Definition 2.6 . An exact frame is a frame which ceases to be one if any of its elements is removed. It can be shown that the notions of Riesz bases, exact frames, and unconditional Schauder bases coincide.

Definition 2.7 .A subset S of $\mathbb{C}^d$ is said to be uniformly separated if

$$\inf_{x,(1+\epsilon_2)\in S, x\neq(1+\epsilon_2)} \|x-y\|_2 > 0.$$

Definition 2. 8 . If $S = (x_k)_k$ is a sequence of real numbers and f_j is a function with S in its domain, then f_S denotes the sequence $(f(x_k))_k$.

III. The multidimensional gradually series over sampling theorem

In [7], Daubechies and DeVore derive the following formula

$$\sum_j f_j\,(t) = \frac{1}{(1+\epsilon_2)} \sum_{(m+\epsilon_0)\in\mathbb{Z}} \sum_j f_j\left(\frac{(m+\epsilon_0)}{(1+\epsilon_2)}\right) g_j\left(t - \frac{(m+\epsilon_0)}{(1+\epsilon_2)}\right), t \in \mathbb{C}, \quad (3)$$

where g_j is infinitely smooth and decays rapidly. Thus over sampling allows the representation of gradually series bandlimited functions as combinations of integer translates of g_j rather than the sinc function. In this sense (3) is a generalization of the WKS theorem. The rapid decay of g_j yields a certain stability in the recovery formula, given bounded perturbations in the sampled data [7].In this section we derive a multidimensional gradually series version of (3)(see[13]).Daubechies and DeVore regard $\mathcal{F}^{-1}(f_j)$ as an element of

$L_2[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]$ for some $\epsilon_2 > 0$. In their proof the obvious fact that

$[-\pi,\pi] \subset [-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]$ allows for the construction of the bump function $\mathcal{F}^{-1}(g_j) \in C^\infty(\mathbb{C})$ which is 1 on $[-\pi,\pi]$ and 0 off $[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]$. If their result is to be generalized to a sampling theorem for PW_E in higher dimensions, a suitable condition for E allowing the existence of a bump function is necessary. If $E \subset \mathbb{C}^d$ is chosen to be compact such that for all $\epsilon_2 > 0, E \subset int((1+\epsilon_2)E)$, then Lemma 8.18 in [9, p. 245], a C^∞-version of the Urysohn lemma, implies the existence of a smooth bump function which is 1 on E and 0 off $(1+\epsilon_2)E$. It is to such regions that we generalize (3) (see [13]) .

Theorem 3.1 . Let $0 \in E \subset \mathbb{C}^d$ be compact such that for all $\epsilon_2 > 0, E \subset int((1+\epsilon_2)E)$. Choose $S = (t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}} \subset \mathbb{C}^d$ such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$, defined by $f_{(m+\epsilon_0)}(\cdot) = e^{i\langle\cdot,t_{(m+\epsilon_0)}\rangle}$, is a gradually series frame for $L_2(E)$ with frame operator S. Let $\epsilon_5 > 0$ with $\mathcal{F}^{-1}(g_j) : \mathbb{C}^d \to \mathbb{C}, \mathcal{F}^{-1}(g_j) \in C^\infty$ where $\mathcal{F}^{-1}(g_j)|_E = 1$ and $\mathcal{F}^{-1}(g_j)|_{((1+\epsilon_2)E)^c} = 0$. If $\epsilon_2 > \epsilon_5 > 0$ and $g_j \in PW_E$, then

$$\sum_j f_j(t) = \frac{1}{(1+\epsilon_2)^d} \sum_{k\in\mathbb{N}} \sum_j \left(\sum_{(m+\epsilon_0)\in\mathbb{N}} B_{k(m+\epsilon_0)}\, f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right) g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right), t \in \mathbb{C}^d, \quad (4)$$

where $B_{k(m+\epsilon_0)} = \langle S^{-1} f_{(m+\epsilon_0)}, S^{-1} f_k\rangle_E$. Convergence of the sum is in $L_2(\mathbb{C}^d)$, hence also uniform. Further, the map $B : \ell_2(N) \to \ell_2(N)$ defined by $(y_k)_{k\in\mathbb{N}} \mapsto \left(\sum_{(m+\epsilon_0)\in\mathbb{N}} B_{k(m+\epsilon_0)}\, y_{(m+\epsilon_0)}\right)_{k\in\mathbb{N}}$ is bounded linear, and is an onto isomorphism if and only if $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$ is a Riesz basis for $L_2(E)$.

Proof . Define $f_{(1+\epsilon_2),(m+\epsilon_0)}(\cdot) = f_{(m+\epsilon_0)}\left(\frac{\cdot}{(1+\epsilon_2)}\right)$. Note that $(f_{(1+\epsilon_2),(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a gradually series frame for $L_2\big((1+\epsilon_2)E\big)$ with frame operator $S_{(1+\epsilon_2)}$.

Step 1*:* We show that

$$\sum_j f_j = \sum_{(m+\epsilon_0)} \sum_j f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\mathcal{F}\left(S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\right)\mathcal{F}^{-1}(g_j)\right], f_j \in PW_E . \quad (5)$$

We know supp$(\mathcal{F}^{-1}(f_j)) \subset E \subset (1+\epsilon_2)E$, so we may work with $\mathcal{F}^{-1}(f_j)$ via its frame decomposition . We have

$$\sum_j \mathcal{F}^{-1}(f_j) = S^{-1}_{(1+\epsilon_2)} S_{(1+\epsilon_2)} \sum_j \left(\mathcal{F}^{-1}(f_j)\right)$$

$$= \sum_{(m+\epsilon_0)} \sum_j \langle \mathcal{F}^{-1}(f_j), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\; S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)},$$

$on\ (1+\epsilon_2)E$. This yields

$$\sum_j \mathcal{F}^{-1}(f_j) = \sum_{(m+\epsilon_0)} \sum_j \langle \mathcal{F}^{-1}(f_j), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E} \left(S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\right)\mathcal{F}^{-1}(g_j), on\ \mathbb{C}^d,$$

since supp$\mathcal{F}(g_j) \subset (1+\epsilon_2)E$. Taking Fourier transforms we obtain

$$\sum_j f_j = \sum_{(m+\epsilon_0)} \sum_j \langle \mathcal{F}^{-1}(f_j), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\, \mathcal{F}\left[\left(S^{-1}_{(1+\epsilon_2)}\right)\mathcal{F}^{-1}(g_j)\right], on\ \mathbb{C}^d. \quad (6)$$

Now

$$\sum_j \langle \mathcal{F}^{-1}(f_j), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E} = \int_{(1+\epsilon_2)E} \sum_j \mathcal{F}^{-1}(f_j)(\xi)\, e^{-i\langle\xi,\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\rangle} d\xi = \sum_j f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)$$

which, when substituted into (6), yields (5).

Step 2: We show that

$$\sum_j f_j(\cdot) = \sum_{(m+\epsilon_0)} \sum_j f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\sum_k \langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, g_j\left(\cdot - \frac{t_k}{(1+\epsilon_2)}\right)\right], (7)$$

where convergence is in L_2.We compute $\mathcal{F}[(S^{-1}_{(\epsilon_2+1)}\, f_{(\epsilon_2+1),(m+\epsilon_0)})\mathcal{F}^{-1}(g_j)]$. For $h \in L_2((1+\epsilon_2)E)$ we have

$$h = S_{(1+\epsilon_2)}(S^{-1}_{(1+\epsilon_2)}h) = \sum_k \langle S^{-1}_{(1+\epsilon_2)}h, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, f_{(1+\epsilon_2),k} = \sum_k \langle h\,, S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\;\, f_{(1+\epsilon_2),k}.$$

Letting $h = S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),(m+\epsilon_0)}$,

$$S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),(m+\epsilon_0)} = \sum_k \langle S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),(m+\epsilon_0)}\,, S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\;\, f_{(1+\epsilon_2),k}.$$

This gives

$$\sum_j \mathcal{F}\big[\big(S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\big)\mathcal{F}^{-1}(g_j)\big](\cdot)$$
$$= \sum_k \sum_j \langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, \mathcal{F}\big[f_{(1+\epsilon_2),k}\mathcal{F}^{-1}(g_j)\big](\cdot)$$
$$= \sum_k \langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\,, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} \int_{(1+\epsilon_2)E} \sum_j e^{i\langle \xi, \frac{t_k}{(1+\epsilon_2)}\rangle}\mathcal{F}^{-1}(g_j)(\xi)e^{-i\langle \xi,\cdot\rangle}\, d\xi$$
$$= \sum_k \langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\,, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} \int_{(1+\epsilon_2)E} \sum_j \mathcal{F}^{-1}(g_j)(\xi)e^{-i\langle \cdot-\frac{t_k}{(1+\epsilon_2)},\xi\rangle}\, d\xi$$
$$= \sum_k \sum_j \langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\,, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, g_j\left(\cdot - \frac{t_k}{(1+\epsilon_2)}\right),$$

so (7) follows from (5).

Step 3: We show that

$$\langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\,, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} = \frac{1}{(1+\epsilon_2)^d}\langle S^{-1}f_{(m+\epsilon_0)}, S^{-1}f_k\rangle_E\ \text{, for}\ \ \epsilon_0 > 0\,, k \in \mathbb{N}. \quad (8)$$

First we show $(S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)})(\cdot) = \frac{1}{(1+\epsilon_2)^d}\, (S^{-1}_{(1+\epsilon_2)} f_{(m+\epsilon_0)})\left(\frac{\cdot}{(1+\epsilon_2)}\right)$, or equivalently that $f_{(1+\epsilon_2),(m+\epsilon_0)} = \frac{1}{(1+\epsilon_2)^d} S_{(1+\epsilon_2)}\left((S^{-1}f_{(m+\epsilon_0)})\left(\frac{\cdot}{(1+\epsilon_2)}\right)\right)$.We have for any $g_j \in L_2((1+\epsilon_2)E)$,

$$\sum_j \langle g_j\,, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} = \int_{(1+\epsilon_2)E} \sum_j g_j(\xi)e^{-i\langle\frac{\xi}{(1+\epsilon_2)},t_k\rangle}\, d\xi$$
$$= (1+\epsilon_2)^d \int_E \sum_j g_j\big((1+\epsilon_2)x\big)e^{-i\langle x,t_k\rangle}\, dx = (1+\epsilon_2)^d \sum_j \langle g_j\big((1+\epsilon_2)(\cdot)\big), f_k\rangle_E.$$

By definition of the frame operator $S_{(1+\epsilon_2)}$, $S_{(1+\epsilon_2)}g_j = \sum_{k\in\mathbb{N}}\langle g_j\,, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\; f_{(1+\epsilon_2),k}$, which then becomes $\sum_j S_{(1+\epsilon_2)}g_j = (1+\epsilon_2)^d \sum_k \sum_j \langle g_j\big((1+\epsilon_2)(\cdot)\big), f_k\rangle_E\, f_{(1+\epsilon_2),k}$. Substituting $g_j = \frac{1}{(1+\epsilon_2)^d}(S^{-1}f_{(m+\epsilon_0)})\left(\frac{\cdot}{(1+\epsilon_2)}\right)$ into the equation above we obtain

$$\frac{1}{(1+\epsilon_2)^d} S_{(1+\epsilon_2)}\left(\big(S^{-1}f_{(m+\epsilon_0)}\big)\left(\frac{\cdot}{(1+\epsilon_2)}\right)\right) = \sum_k \langle S^{-1}f_{(m+\epsilon_0)}, f_k\rangle_E\, f_{(1+\epsilon_2),k}$$
$$= \big(S\big(S^{-1}f_{(m+\epsilon_0)}\big)\big)\left(\frac{\cdot}{(1+\epsilon_2)}\right) = f_{(1+\epsilon_2),(m+\epsilon_0)}.$$

We now compute the desired inner product:

$$\langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}$$
$$= \frac{1}{(1+\epsilon_2)^{2d}} \int_{(1+\epsilon_2)E} \big(S^{-1}f_{(m+\epsilon_0)}\big)\left(\frac{x}{(1+\epsilon_2)}\right)\overline{(S^{-1}f_k)\left(\frac{x}{(1+\epsilon_2)}\right)}\, dx$$
$$= \frac{(1+\epsilon_2)^d}{(1+\epsilon_2)^{2d}} \int_E \big(S^{-1}f_{(m+\epsilon_0)}\big)(x)\overline{(S^{-1}f_k)(x)}\, dx = \frac{1}{(1+\epsilon_2)^d}\langle S^{-1}f_{(m+\epsilon_0)}, S^{-1}f_k\rangle_E\,.$$

Note that (7) becomes

$$\sum_j f_j(\cdot) = \frac{1}{(1+\epsilon_2)^d} \sum_{(m+\epsilon_0)} \sum_j f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\sum_k \langle S^{-1}f_{(m+\epsilon_0)}, S^{-1}f_k\rangle g_j\left(\cdot - \frac{t_k}{(1+\epsilon_2)}\right)\right]. \quad (9)$$

Step4: The map $V : \ell_2(\mathbb{N}) \mapsto \ell_2(\mathbb{N})$ given by $x = (x_k)_{k\in\mathbb{N}} \mapsto \left(\sum_{(m+\epsilon_0)} B_{k(m+\epsilon_0)}x_{(m+\epsilon_0)}\right)_{k\in\mathbb{N}} = Bx$ is bounded linear and self-adjoint. Let $(d_k)_{k\in\mathbb{N}}$ be the standard basis for $\ell_2(\mathbb{N})$, and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis for $L_2(E)$. Then

$$Vd_j = (B_{kj})_{k\in\mathbb{N}} = \sum_k B_{kj}\, d_k = \sum_k \langle S^{-1}f_j, S^{-1}f_k\rangle_{d_k} = \sum_k \langle L^*(S^{-1})^2 Le_j, e_k\rangle_{d_k},$$

where L is the synthesis f operator, i.e., $S = LL^*$. Define $\varphi : \ell^2(\mathbb{N}) \mapsto L_2(E)$ by $\varphi(d_k) = e_k, k \in \mathbb{N}$. Clearly φ is unitary. It follows that $V = \varphi^{-1}L^*(S^{-1})^2 L\varphi$, which concludes Step 4. From here on we identify V with B. Clearly B is an onto isomorphism if and only if L and L^* are both onto, i.e., if and only if the map $Le_{(m+\epsilon_0)} = f_{(m+\epsilon_0)}$ is an onto isomorphism.

Step 5: Verification of (4). Recalling Definition 3.8, $f_{S_j/(1+\epsilon_2)} = \left(f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right)_{(m+\epsilon_0)\in\mathbb{N}}$, for each $t \in \mathbb{C}^d$, let $g_{(1+\epsilon_2)_j}(t) = \left(g_j\left(t - \frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right)_{(m+\epsilon_0)\in\mathbb{N}}$. Noting that $f_j\left(\frac{\cdot}{(1+\epsilon_2)}\right), g_j\left(t - \frac{\cdot}{(1+\epsilon_2)}\right) \in L_2((1+\epsilon_2)E)$, and recalling that $(f_{(1+\epsilon_2),(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a frame for $L_2((1+\epsilon_2)E)$, we have

$$\sum_{(m+\epsilon_0)}\sum_j \left| f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right|^2 = \sum_{(m+\epsilon_0)}\sum_j \left|\langle \mathcal{F}^{-1}(f_j), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\right|^2 \le A_{(1+\epsilon_2)}\sum_j \left\|\mathcal{F}^{-1}(f_j)\right\|^2, \quad (10)$$

and

$$\sum_{(m+\epsilon_0)}\sum_j \left| g_j\left(t - \frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right|^2 = \sum\sum_j \left|\left\langle \mathcal{F}^{-1}\left(g_j\left(t - \frac{\cdot}{(1+\epsilon_2)}\right)\right), f_{(1+\epsilon_2),(m+\epsilon_0)}\right\rangle_{(1+\epsilon_2)E}\right|^2$$
$$\le A_{(1+\epsilon_2)}\sum_j \left\|\mathcal{F}^{-1}\left(g_j\left(t - \frac{\cdot}{(1+\epsilon_2)}\right)\right)\right\|^2 .$$

Note that (9) becomes

$$\sum_j f_j(t) = \frac{1}{(1+\epsilon_2)^d}\sum_{(m+\epsilon_0)}\sum_j f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\sum_k\sum_j B_{k(m+\epsilon_0)}\, g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)\right]$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_{(m+\epsilon_0)}\sum_j f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\overline{\left[\sum_k\sum_j B_{(m+\epsilon_0)k}\,\overline{g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)}\right]}$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_{(m+\epsilon_0)}\sum_j \left(f_{S/(1+\epsilon_2)}\right)_{(m+\epsilon_0)}\left(B\overline{g_{j\,(1+\epsilon_2)}(t)}\right)_{(m+\epsilon_0)} = \frac{1}{(1+\epsilon_2)^d}\sum_j \langle f_{j\,\frac{S}{(1+\epsilon_2)}}, B\overline{g_{j\,(1+\epsilon_2)}(t)}\rangle$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_j \langle Bf_{S/(1+\epsilon_2)}, \overline{g_{j\,(1+\epsilon_2)}(t)}\rangle$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_k\sum_j \left(Bf_{S/(1+\epsilon_2)}\right)_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_{k\in\mathbb{N}}\sum_j\left(\sum_{(m+\epsilon_0)\in\mathbb{N}} B_{k(m+\epsilon_0)} f_j\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right) g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right),$$

which proves (4).

Step 6: We verify that convergence in (4) is in $L_2(\mathbb{C})$ (hence uniform). Define

$$f_{(m+\epsilon_0)}(t) = \frac{1}{(1+\epsilon_2)^d}\sum_{1\le k\le (m+\epsilon_0)}\sum_j \left(Bf_{S/(1+\epsilon_2)}\right)_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$

and

$$f_{m,(m+\epsilon_0)}(t) = \frac{1}{(1+\epsilon_2)^d}\sum_{m\le k\le (m+\epsilon_0)}\sum_j \left(Bf_{S/(1+\epsilon_2)}\right)_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right).$$

Then

$$[\mathcal{F}^{-1}(f_{m,(m+\epsilon_0)})](\xi) = \frac{1}{(1+\epsilon_2)^d}\sum_{m\le k\le (m+\epsilon_0)}\sum_j \left(Bf_{S/(1+\epsilon_2)}\right)_k \mathcal{F}^{-1}\left[g_j\left(. - \frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right]$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_{m\le k\le (m+\epsilon_0)}\sum_j \left(Bf_{S/(1+\epsilon_2)}\right)_k \mathcal{F}^{-1}(g_j)(\xi)e^{i\langle \xi, \frac{t_k}{(1+\epsilon_2)}\rangle},$$

so

$$\|[\mathcal{F}^{-1}(f_{m,(m+\epsilon_0)})]\|_2^2 = \frac{1}{(1+\epsilon_2)^d} \int_{(1+\epsilon_2)E} \sum_j |\mathcal{F}^{-1}(g_j)(\xi)|^2 \left| \sum_{m\le k\le (m+\epsilon_0)} (Bf_{S/(1+\epsilon_2)})_k e^{i\langle \xi, \frac{t_k}{(1+\epsilon_2)}\rangle} \right|^2 d\xi$$
$$\le \frac{1}{(1+\epsilon_2)^d} \left\| \sum_{m\le k\le (m+\epsilon_0)} (Bf_{S/(1+\epsilon_2)})_k f_{(1+\epsilon_2),k} \right\|_2^2.$$

If $(h_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a orthonormal basis for $L_2((1+\epsilon_2)E)$, then the map $Th_K = f_{(1+\epsilon_2),k}$ (the synthesis operator) is bounded linear, so

$$\|[\mathcal{F}^{-1}(f_{m,(m+\epsilon_0)})]\|_2^2 \le \frac{1}{(1+\epsilon_2)^d} \left\| T\left(\sum_{m\le k\le (m+\epsilon_0)} (Bf_{S/(1+\epsilon_2)})_k h_K \right) \right\|_2^2$$
$$\le \frac{1}{(1+\epsilon_2)^d} \|T\|^2 \sum_{m\le k\le (m+\epsilon_0)} \left| (Bf_{S/(1+\epsilon_2)})_k \right|^2.$$

But $Bf_{S/(1+\epsilon_2)} \in \ell^2(\mathbb{N})$, so $\|[\mathcal{F}^{-1}(f_{m,(m+\epsilon_0)})]\|2 \to 0$ as $m \to \infty, \epsilon_0 > 0$. As $\mathcal{F}^{-1}$ is an onto isomorphism, we have $\|f_{m,(m+\epsilon_0)}\| \to 0$, implying that $\|f - f_{(m+\epsilon_0)}\| \to 0$ as $m \to \infty$. Note that (3) is conveniently written as

$$\sum_j f_j(t) = \frac{1}{(1+\epsilon_2)^d} \sum_k \sum_j (Bf_{S/(1+\epsilon_2)})_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right), t \in \mathbb{C}^d. \quad (11)$$

Remark . There is a geometric characterization of sets $E \subset \mathbb{C}^d$ such that $E \subset int((1+\epsilon_2)E)$ for all $\epsilon_2 > 0$. Intuitively, E must be a continuous radial stretching of the closed unit ball". This is precisely formulated in the following proposition .

Proposition 3.2 . If $0 \in E \subset \mathbb{C}^d$ is compact, then the following are equivalent:

(i) $E \subset int((1+\epsilon_2)E)$ for all $\epsilon_2 > 0$.

(ii) There exists a continuous map $\varphi : S^{d-1} \to (0,\infty)$ such that $E = \{ty\varphi(y) \mid y \in S^{d-1}, t \in [0,1]\}$. The following is a simplified version of Theorem 3.1 , which is proven in a similar fashion (see [13]):

Theorem 3.3 . Choose $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}} \subset \mathbb{C}^d$ such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$, defined by $f_{(m+\epsilon_0)}(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, t_{(m+\epsilon_0)}\rangle}$, is a frame for $L_2([-\pi,\pi]^d)$. If $f \in PW_E$,then

$$\sum_j f_j(t) = \sum_{k\in\mathbb{N}} \left(\sum_{(m+\epsilon_0)\in\mathbb{N}} \sum_j B_{k(m+\epsilon_0)} f_j\left(t_{(m+\epsilon_0)}\right) \right) SINC(\pi(t-t_k)), t \in \mathbb{C}^d. \quad (12)$$

The matrix B and the convergence of the sum are as in Theorem 3.1.Then (4) generalizes (12) in the same way that (3) generalizes the WKS equation. We can write (12) as

$$\sum_j f_j(t) = \sum_{k\in\mathbb{N}} \sum_j \left(Bf_{S_j}\right)_k SINC(\pi(t-t_k)). \quad (13)$$

The preceding result is similar in spirit to Theorem 1.9 in [4, p. 19].Gradually series frames for$L_2(E)$ satisfying the conditions in Theorems 3.9 and 5.2 occur in abundance. The following result is due to Beurling in
[5, see Theorem 1, Theorem 2, and (20)].

Theorem 3.4 . Let $\Lambda \subset \mathbb{C}^d$ be countable such that

$$r(\Lambda) = \frac{1}{2} \inf_{(1+\epsilon_2),\mu\in\Lambda,(1+\epsilon_2)\neq\mu} \|(1+\epsilon_2)-\mu\|_2 > 0 \text{ and } \mathbb{C}(\Lambda) = \sup_{\xi\in\mathbb{C}^d} \inf_{(1+\epsilon_2)\in\Lambda} \|(1+\epsilon_2) - \mu\|_2 < \frac{\pi}{2}.$$

If E is a subset of the closed unit ball in $\mathbb{C}^d$ and E has positive measure, then $\{e^{i\langle\cdot,(1+\epsilon_2)\rangle} \mid (1+\epsilon_2) \in \Lambda\}$ is a gradually series frame for $L_2(E)$.

IV. REMARKS REGARDING THE STABILITY OF THEOREM 3.1

A desirable trait in a recovery formula is stability given error in the sampled data (see [13]). Suppose we have sample values $\tilde{f}_{j\,(m+\epsilon_0)} = f_j\left(\frac{(m+\epsilon_0)}{(1+\epsilon_2)}\right) + \epsilon_{(m+\epsilon_0)}$ where $sup_{(m+\epsilon_0)}|\epsilon_{(m+\epsilon_0)}| = \epsilon$.

If in (3) we replace $f\left(\frac{(m+\epsilon_0)}{(1+\epsilon_2)}\right)$ *by* $\tilde{f}_{(m+\epsilon_0)}$, and call the resulting expression $\tilde{f}$, then we have

$$\sum_j |f_j(t) - \tilde{f}_j(t)| \le \frac{\epsilon}{(1+\epsilon_2)} \sum_{(m+\epsilon_0)\in\mathbb{Z}} \sum_j \left| g_j\left(t - \frac{(m+\epsilon_0)}{(1+\epsilon_2)}\right) \right| \le \frac{\epsilon}{(1+\epsilon_2)} \sum_j \|g_j{}'\|_{L_1} + \sum_j \epsilon \|g_j\|_{L_1}.$$

It follows that (3) is certainly stable under ℓ_∞ perturbations in the data, while the WKS sampling theorem is not. For a more detailed discussion see [7]. Such a stability result is not immediately forthcoming for (4), as the following example illustrates. Restricting to $d = 1$, let $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}}$ satisfy $t_2 = D \notin \mathbb{Z}$, and $t_{(m+\epsilon_0)} = (m+\epsilon_0)$ for $(m+\epsilon_0) \neq 0$. The forthcoming discussion in Section 5 shows that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$.

Note that when $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a Riesz basis, the sequence $\left(S^{-1}f_{(m+\epsilon_0)}\right)_{(m+\epsilon_0)}$ is its biorthogonal sequence.

The matrix B associated to this basis is computed as follows. The biorthogonal functions $(G_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}}$ for $(sinc(\pi(\cdot - (m+\epsilon_0))))_{(m+\epsilon_0)\in\mathbb{Z}}$ are $G_{(m+\epsilon_0)}(t) = \frac{(-1)^{(m+\epsilon_0)}(m+\epsilon_0)(t-D)sinc(\pi t)}{((m+\epsilon_0)-D)(t-(m+\epsilon_0))}, (m+\epsilon_0) \neq 0,$

And $G_0(t) = \frac{sinc(\pi t)}{sinc(\pi D)}$. That these functions are in $PW_{[-\pi,\pi]}$ is verified by applying the Paley–Wiener theorem [1, p. 85], and the biorthogonality condition is verified by applying (1). Again using (1), we obtain

(i) $B_{m0} = \langle G_0, G_m\rangle = \frac{D(-1)^m}{sinc(\pi D)(m-D)}, m \neq 0,$

(ii) $B_{00} = \langle G_0, G_0\rangle = \frac{1}{sinc^2(\pi D)},$

(iii) $B_{m(m+\epsilon_0)} = \langle G_{(m+\epsilon_0)}, G_m\rangle = \delta_{(m+\epsilon_0)m} + \frac{D^2(-1)^{(m+\epsilon_0)+m}}{((m+\epsilon_0)-D)(m-D)}, else.$

Note that the rows of B are not in ℓ_1, so that as an operator acting on ℓ_∞, B does not act boundedly. Consequently, the equation

$$\sum_j \tilde{f}_j(t) = \frac{1}{(1+\epsilon_2)}\sum_k\sum_j \left(B\tilde{f}_{S/(1+\epsilon_2)}\right)_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right) \tag{14}$$

is not defined for all perturbed sequences $\tilde{f}_{S/(1+\epsilon_2)}$ where $\left(\tilde{f}_{S/(1+\epsilon_2)}\right)_{(m+\epsilon_0)} = \left(f_{S/(1+\epsilon_2)}\right)_{(m+\epsilon_0)} + \epsilon_{(m+\epsilon_0)}$ where $\sup_{(m+\epsilon_0)}\left|\epsilon_{(m+\epsilon_0)}\right| = \epsilon$. Despite the above failure, the following shows that there is some advantage of (4) over (2).If $\tilde{f}_{S/(1+\epsilon_2)}$ is some perturbation of $f_{S/(1+\epsilon_2)}$ such that $\left\|B\tilde{f}_{S/(1+\epsilon_2)} - Bf_{S/(1+\epsilon_2)}\right\|_\infty \leq \epsilon$, then

$$\begin{aligned}\sup_{\xi\in\mathbb{C}^d}\sum_j\left|f_j(t) - \tilde{f}_j(t)\right| &= \sup_{\xi\in\mathbb{C}^d}\left|\frac{1}{(1+\epsilon_2)}\sum_k\sum_j\left(B\left(f_{S/(1+\epsilon_2)} - \tilde{f}_{S/(1+\epsilon_2)}\right)\right)_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)\right| \\ &\leq \epsilon\sup_{\xi\in\mathbb{C}^d}\frac{1}{(1+\epsilon_2)}\sum_k\sum_j\left|g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)\right| \leq M \end{aligned} \tag{15}$$

V. RESTRICTION OF THE SAMPLING THEOREM TO THE CASE WHERE THE EXPONENTIAL GRADUATE FRAME IS A RIESZ BASIS

From here on (see [13]), we focus on the case where $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$ is an ℓ_∞ perturbation of the lattice $\mathbb{Z}^d$, and $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$. In this case, under the additional constraint that the sample nodes are asymptotically the integer lattice, the following theorem gives a computationally feasible version of (4). The summands in (4) involves an infinite invertible matrix B, though under the constraints mentioned above, we show that B can be replaced by a related finite-rank operator which can be computed concretely. Precisely, one has the following (see [13]).

Theorem 5.1 .Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and $S = (t_k)_{k\in\mathbb{N}} \subset \mathbb{C}^d$ such that $\lim_{k\to\infty}\|(m+$ $\epsilon 0k - tk\infty = 0$.

Define $e_k, f_k : \mathbb{C}^d \to \mathbb{C}$ *by* $e_k(x) = \frac{1}{(2\pi)^{d/2}}e^{i\langle(m+\epsilon_0)_k,\ x\rangle}$ and $\frac{1}{(2\pi)^{d/2}}e^{i\langle t_k,\ x\rangle}$, and let $(h_k)_k$ be the standard basis for $\ell_2(\mathbb{N})$. *Let* $P_l: \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the orthogonal projection onto span$\{h_1,\ldots,h_l\}$. If $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, then for all $f \in PW_{[-\pi,\pi]^d}$, we have

$$\sum_j f_j(t) = \lim_{l\to\infty}\frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\left[(P_lB^{-1}P_l)^{-1}f_{\frac{S}{(1+\epsilon_2)}}\right]_k\sum_j g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right), t \in \mathbb{C}^d, \tag{16}$$

where convergence is in L_2 and uniform. Furthermore,

$$(P_lB^{-1}P_l)_{(m+\epsilon_0)m} = \begin{cases} sinc\pi(t_{(m+\epsilon_0),1} - t_{m,1}) \cdots sinc\pi(t_{(m+\epsilon_0),d} - t_{m,d}), & 1 \leq (m+\epsilon_0), m \leq l, \\ 0, & otherwise. \end{cases}$$

Convergence of the sum is in L_2 and also uniform.

The matrix $P_lB^{-1}P_l$ is clearly not invertible as an operator on ℓ_2, and it should be interpreted as the inverse of an $l \times l$ matrix acting on the first l coordinates of $f_{S/(1+\epsilon_2)}$. The following version of Theorem 5.1 avoids over sampling . Its proof is similar to that of Theorem 5.1.

Theorem 5.2 . Under the hypotheses of Theorem 5.1 ,

$$\sum_j f_j(t) = \lim_{l\to\infty} \sum_{k=1}^{l} [(P_l B^{-1} P_l)^{-1} f_S]_k \; SINC(t - t_k), t \in \mathbb{C}^d, \quad (17)$$

where convergence of the sum is both L_2 and uniform. The following lemma forms the basis of the proof of the preceding theorems, as well as the other results in the paper (see [13]) .

Lemma 5.3 . Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{C}^d$. Define $e_k, f_k : \mathbb{C}^d \to \mathbb{C}$ by $e_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle (m+\epsilon_0)_k, x\rangle}$ and $f_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle t_k, x\rangle}$. Then for any $\epsilon_4 > \epsilon_3 \geq 0$, and any finite sequence $(a_k)_{k=(1+\epsilon_3)}^{(1+\epsilon_4)}$, we have

$$\left\| \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \left(\frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{\,i\langle(\cdot),(m+\epsilon_0)_k\rangle} - \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{\,i\langle(\cdot),t_k\rangle} \right) \right\|_2$$

$$\leq \left(e^{\pi d\,(sup_{\,(1+\epsilon_3)\leq k\leq(1+\epsilon_4)} \|(m+\epsilon_0)_k - t_k\|_\infty)} - 1 \right) \left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2 \right)^{1/2} . \quad (18)$$

Proof . Let $\delta_k = t_k - (m+\epsilon_0)_k$ where $\delta_k = (\delta_{k_1}, \ldots, \delta_{k_d})$. Then

$$\varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x) = \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \frac{a_k}{(2\pi)^{\frac{d}{2}}} \left[e^{i\langle (m+\epsilon_0)_k, x\rangle}, e^{i\langle t_k, x\rangle} \right] = \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle (m+\epsilon_0)_k, x\rangle} \left[1 - e^{i\langle \delta_k, x\rangle} \right] . (19)$$

Now for any $\boldsymbol{\delta_k}$,

$$\begin{aligned} 1 - e^{i\langle \delta_k, x\rangle} &= 1 - e^{i\delta_{k_1} x_1} \ldots e^{i\delta_{k_d} x_d} \\ &= 1 - \left(\sum_{j_1=0}^{\infty} \frac{(i\delta_{k_1} x_1)^{j_1}}{j_1!} \right) \ldots \left(\sum_{j_d=0}^{\infty} \frac{(i\delta_{k_d} x_d)^{j_d}}{j_d!} \right) \\ &= 1 - \sum_{\substack{(j_1,\ldots,j_d) \\ j_i \geq 0}} \frac{(i\delta_{k_1} x_1)^{j_1} \cdot\cdot\cdot\cdot\cdot (i\delta_{k_d} x_d)^{j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} \\ &= - \sum_{(j_1,\ldots,j_d)\in J} i^{j_1,\ldots,j_d} \frac{(i\delta_{k_1} x_1)^{j_1} \cdot\cdot\cdot\cdot\cdot (i\delta_{k_d} x_d)^{j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} , \end{aligned}$$

where $J = \{(j_1, \ldots, j_d) \in \mathbb{Z}^d \mid j_i \geq 0, (j_1, \ldots, j_d) \neq 0\}$. Then (19) becomes

$$\begin{aligned} \varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x) &= - \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle (m+\epsilon_0)_k, x\rangle} \left[\sum_{(j_1,\ldots,j_d)\in J} i^{j_1,\ldots,j_d} \frac{(i\delta_{k_1} x_1)^{j_1} \cdot\cdot\cdot\cdot\cdot (i\delta_{k_d} x_d)^{j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} \right] \\ &= - \sum_{(j_1,\ldots,j_d)\in J} \frac{x_1^{j_1} \cdot\cdot\cdot x_d^{j_d}}{j_1! \cdot\cdot\cdot j_d!} i^{j_1,\ldots,j_d} \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \frac{a_k}{(2\pi)^{\frac{d}{2}}} \delta_{k_1}^{j_1} \cdot\cdot\cdot\cdot\cdot \delta_{k_d}^{j_d} \, e^{i\langle (m+\epsilon_0)_k, x\rangle} , \end{aligned}$$

so

$$\left| \varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x) \right| \leq \sum_{(j_1,\ldots,j_d)\in J} \frac{\pi^{j_1,\ldots,j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} a_k \delta_{k_1}^{j_1} \cdot\cdot\cdot\cdot\cdot \delta_{k_d}^{j_d} \frac{e^{i\langle (m+\epsilon_0)_k, x\rangle}}{(2\pi)^{\frac{d}{2}}} .$$

For brevity denote the outer summand above by $h_{\boldsymbol{j_1,\ldots,j_d}}(t)$. Then

$$\begin{aligned} \left(\int_{[-\pi,\pi]^d} \left| \varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x) \right|^2 dt \right)^{\frac{1}{2}} &\leq \left(\int_{[-\pi,\pi]^d} \left| \sum_{(j_1,\ldots,j_d)\in J} h_{j_1,\ldots,j_d}(x) \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \sum_{(j_1,\ldots,j_d)\in J} \left(\int_{[-\pi,\pi]^d} \left| h_{j_1,\ldots,j_d}(x) \right|^2 dx \right)^{\frac{1}{2}} , \end{aligned}$$

so that

$$\|\varphi_{(1+\epsilon_3),(1+\epsilon_4)}\|_2 \le \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdots\ j_d!}\left(\int_{[-\pi,\pi]^d}\left|\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} a_k\delta_{k_1}^{j_1}\ \cdots\ \delta_{k_d}^{j_d}\ \frac{e^{i\langle(m+\epsilon_0)_k,x\rangle}}{(2\pi)^{\frac{d}{2}}}\right|^2 dx\right)^{\frac{1}{2}}$$

$$= \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdots\cdots\ j_d!}\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\left|\delta_{k_1}^{j_1}\right|^2\ \cdots\cdots\ \left|\delta_{k_d}^{j_d}\right|^2\right)^{\frac{1}{2}}$$

$$\le \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdots\ j_d!}\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\left(\sup_{(1+\epsilon_3)\le k\le(1+\epsilon_4)}\|(m+\epsilon_0)_k - t_k\|_\infty\right)^{2(j_1,\dots,j_d)}\right)^{\frac{1}{2}}$$

$$= \sum_{(j_1,\dots,j_d)\in J} \frac{\pi\left(sup_{(1+\epsilon_3)\le k\le(1+\epsilon_4)}\|(m+\epsilon_0)_k - t_k\|_\infty\right)^{j_1,\dots,j_d}}{j_1!\ \cdots\cdots\ j_d!}\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\right)^{\frac{1}{2}}$$

$$= \left[\prod_{l=1}^{d}\left(\sum_{j_\ell=0}^{\infty}\frac{\pi\left(sup_{(1+\epsilon_3)\le k\le(1+\epsilon_4)}\|(m+\epsilon_0)_k - t_k\|_\infty\right)^{j_\ell}}{j_\ell!}\right)-1\right]\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\right)^{\frac{1}{2}}$$

$$= \left(e^{\pi d\left(sup_{(1+\epsilon_3)\le k\le(1+\epsilon_4)}\|(m+\epsilon_0)_k - t_k\|_\infty\right)} - 1\right)\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\right)^{\frac{1}{2}}.$$

Corollary 5.4 . Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{C}^d$ such that $\sup_{k\in\mathbb{N}}\|(m+\epsilon_0)_k - t_k\|_\infty = L < \infty$. Define $e_k, f_k : \mathbb{C}^d \to \mathbb{C}$ by $e_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle(m+\epsilon_0)_k,x\rangle}$ and $f_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle t_k,x\rangle}$.Then the map $T: L_2[-\pi,\pi]^d \to L_2[-\pi,\pi]^d$, defined by $Te_{(m+\epsilon_0)} = e_{(m+\epsilon_0)} - f_{(m+\epsilon_0)}$, satisfies the following estimate:

$$\|T\| \le e^{\pi L d} - 1\,. \tag{20}$$

Proof . Lemma 5.3 shows that T is uniformly continuous on a dense subset of the ball in $L_2(E)$, so T is bounded on $L_2[-\pi,\pi]^d$. The inequality (20) follows immediately.

Corollary 5. 5 . Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}} \subset \mathbb{C}^d$, and let e_k, f_k and T be defined as in Corollary 5.4. For each $l \in \mathbb{N}$, define T_l by $T_{le_k} = e_k - f_k$ for $1 \le k \le l$, and $T_{le_k} = 0$ for $l < k$. If $\lim_{k\to\infty}\|(m+\epsilon_0)_k - t_k\|_\infty = 0$, then $\lim_{l\to\infty} T_l = T$ in the operator norm. In particular, T is a compact operator.

Proof . As

$$(T - T_l)\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} a_k(e_k - f_k) - \sum_{k=1}^{l} a_k(e_k - f_k)$$

$$= \sum_{k=l+1}^{\infty} a_k(e_k - f_k) = T\left(\sum_{k=l+1}^{\infty} a_k e_k\right),$$

the estimate derived in Lemma 5.3 yields

$$\left\|(T - T_l)\left(\sum_{k=1}^{\infty} a_k e_k\right)\right\|_2 = \left\|T\left(\sum_{k=l+1}^{\infty} a_k e_k\right)\right\|_2$$

$$\le \left(e^{\pi d\, sup_{k\ge l+1}\|\delta_k\|_\infty} - 1\right)\left\|\sum_{k=1}^{\infty} a_k e_k\right\|_2,$$

so $\|(T - T_l)\|_2 \to 0$ as $l \to \infty$. As T_l has finite rank, we deduce (see [13]) that T is compact .

The following proof due to [13] .

proof of Theorem 5.1 . Step 1*:* B is a compact perturbation of the identity map, namely

$$B = I + \lim_{l\to\infty}(-P_l + (P_l B^{-1} P_l)^{-1}). \tag{21}$$

Since $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, $L^* = (I - T)$ is an onto isomorphism where $T_{e_k} = e_k - f_k$; so B simplifies to $(I - T)^{-1}(I - T^*)^{-1}$. We examine $B^{-1} = (I - T^*)(I - T) = I + (T^*T - T - T^*) = I + \Delta$,where Δ is a compact operator. If an operator $\Delta: H \to H$ is compact then so is Δ^*, hence $P_l\Delta P_l \to \Delta$ in the operator norm because

$$\|P_l\Delta P_l - \Delta\| \le \|P_l\Delta P_l - P_l\Delta\| + \|P_l\Delta - \Delta\| \le \|\Delta P_l - \Delta\| + \|P_l\Delta - \Delta\| = \|P_l\Delta^* - \Delta^*\| + \|P_l\Delta - \Delta\| \to 0 \ .$$

We have $B^{-1} = \lim_{l\to\infty}(I + P_l\Delta P_l) = \lim_{l\to\infty}(I + P_l(B^{-1} - I)P_l) = \lim_{l\to\infty}(I - P_l + P_lB^{-1} P_l)$.

Now $(P_lB^{-1} P_l)$ restricted to the first l rows and columns is the Grammian matrix for the set $(f_1,\dots, f_l)$ which can be shown (in a straightforward manner) to be linearly independent. We conclude that $P_lB^{-1} P_l$ is invertible as an $l \times l$ matrix. By $(P_lB^{-1} P_l)^{-1}$ we mean the inverse as an $l \times l$ matrix and zeroes elsewhere. Observing that the ranges of $P_lB^{-1} P_l$ and $(P_lB^{-1} P_l)^{-1}$are in the kernel of $1 - P_l$, and that the range of $I - P_l$ is in the kernels of $P_lB^{-1} P_l$ and $(P_lB^{-1} P_l)^{-1}$, we easily compute

$$(I - P_l + (P_lB^{-1} P_l)^{-1})^{-1} = I - P_l + P_lB^{-1} P_l,$$

so that $B^{-1} = \lim_{l\to\infty}(I - P_l + (P_lB^{-1} P_l)^{-1})^{-1}$, implying

$$B = \lim_{l\to\infty}(I - P_l + (P_lB^{-1} P_l)^{-1}) = \lim_{l\to\infty} B_l = \lim_{l\to\infty}(- P_l + (P_lB^{-1} P_l)^{-1}) \ .$$

Step 2: We verify (16) and its convergence properties. Recalling (11), we have

$$\sum_j f_j(t) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\sum_j\left[(I - P_l + (P_lB^{-1} P_l)^{-1})f_{S/(1+\epsilon_2)}\right]_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$
$$= \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\sum_j\left[(B - B_l)\, f_{S/(1+\epsilon_2)}\right]_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$

implying

$$\sum_j f_j(t) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\sum_j\left[(P_lB^{-1} P_l)^{-1}f_{S/(1+\epsilon_2)}\right]_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right) =$$
$$\frac{1}{(1+\epsilon_2)^d}\sum_j\sum_{k=1}^{\infty}\left[(B - B_l)f_{S/(1+\epsilon_2)}\right]_k g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$
$$+ \frac{1}{(1+\epsilon_2)^d}\sum_{k=l+1}^{\infty}\sum_j f_j\left(\frac{t_k}{(1+\epsilon_2)}\right) g_j\left(t - \frac{t_k}{(1+\epsilon_2)}\right).$$

Therefore,

$$\left\|\sum_j f_j(.) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\sum_j\left[(P_lB^{-1} P_l)^{-1}f_{S/(1+\epsilon_2)}\right]_k g_j\left(. - \frac{t_k}{(1+\epsilon_2)}\right)\right\|_2$$
$$= \left\|\frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\sum_j\left[(B - B_l)\, f_{S/(1+\epsilon_2)}\right]_k g_j\left(. - \frac{t_k}{(1+\epsilon_2)}\right) + \frac{1}{(1+\epsilon_2)^d}\sum_{k=l+1}^{\infty}\sum_j f_j\left(\frac{t_k}{(1+\epsilon_2)}\right) g_j\left(. - \frac{t_k}{(1+\epsilon_2)}\right)\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$
$$= \frac{1}{(1+\epsilon_2)^d}\left\|\mathcal{F}^{-1}(g_j)(.)\left(\sum_{k=1}^{\infty}\left[(B - B_l)f_{S/(1+\epsilon_2)}\right]_k e^{i\langle .\,,\frac{t_k}{(1+\epsilon_2)}\rangle} + \sum_{k=l+1}^{\infty}\sum_j f_j\left(\frac{t_k}{(1+\epsilon_2)}\right) e^{i\langle .\,,\frac{t_k}{(1+\epsilon_2)}\rangle}\right)\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$

after taking the inverse Fourier transform. Now

$$\left\|\sum_j f_j\,(.) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\sum_j\left[(P_lB^{-1} P_l)^{-1}f_{S/(1+\epsilon_2)}\right]_k g_j\left(. - \frac{t_k}{(1+\epsilon_2)}\right)\right\|_2$$
$$\le \frac{1}{(1+\epsilon_2)^d}\left\|\sum_{k=1}^{\infty}\left[(B - B_l)f_{S/(1+\epsilon_2)}\right]_k e^{i\langle .\,,\frac{t_k}{(1+\epsilon_2)}\rangle}\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$
$$+ \frac{1}{(1+\epsilon_2)^d}\left\|\sum_{k=l+1}^{\infty}\sum_j f_j\left(\frac{t_k}{(1+\epsilon_2)}\right) e^{i\langle .\,,\frac{t_k}{(1+\epsilon_2)}\rangle}\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$

$$\leq \frac{M}{(1+\epsilon_2)^d} \|(B - B_l) f_{S/(1+\epsilon_2)}\|_{\ell^2(\mathbb{N})} + \frac{M}{(1+\epsilon_2)^d} \left(\sum_{k=l+1}^{\infty} \sum_j \left| f_j \left(\frac{t_k}{(1+\epsilon_2)} \right) \right|^2 \right)^{\frac{1}{2}},$$

since $\left(f_k \left(\frac{\cdot}{(1+\epsilon_2)} \right) \right)_k$ is a Riesz basis for $L_2[-(1+\epsilon_2)\pi, (1+\epsilon_2)\pi]^d$. Since $B_l \to B$ as $l \to \infty$ and $\left(f_j \left(\frac{t_k}{(1+\epsilon_2)} \right) \right)_k \in \ell^2(\mathbb{N})$, the last two terms in the inequality above tend to zero, which proves the required result. Finally, to compute $(P_l B^{-1} P_l)_{nm}$, recall that $B^{-1} = (I - T^*)(I - T)$. Proceeding in a manner similar to the proof of (10), we obtain

$$B^{-1}_{m(m+\epsilon_0)} = \langle LL^* e_{(m+\epsilon_0)}, e_m \rangle = \langle L^* e_{(m+\epsilon_0)}, L^* e_m \rangle = \langle f_{(m+\epsilon_0)}, f_m \rangle$$
$$= sinc\pi\left(t_{(m+\epsilon_0),1} - t_{m,1}\right) \cdot \cdots \cdot sinc\pi\left(t_{(m+\epsilon_0),d} - t_{m,d}\right).$$

The entries of $P_l B^{-1} P_l$ agree with those of B^{-1} when $1 \leq (m+\epsilon_0), m \leq l$.

One generalization of Kadec's 1/4 theorem given by Pak and Shin in [11] (which is actually a special case of Avdonin's theorem) is:

Theorem 5.6 . Let $(t_{(m+\epsilon_0)})_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence of distinct points such that

$$\lim_{|(m+\epsilon_0)|\to\infty} sup|(m+\epsilon_0) - t_{(m+\epsilon_0)}| = L < \frac{1}{4}.$$

Then the sequence of functions $(f_k)_{k\in\mathbb{Z}}$, defined by $f_k(x) = \frac{1}{\sqrt{2\pi}} e^{it_k x}$,
is a Riesz basis for $L_2[-\pi,\pi]$. Theorem 5.6 shows that in the univariate case of Theorem 5.1 the restriction that $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]$can be dropped. The following example shows that the multivariate case is very different. Let $\left(e_{(m+\epsilon_0)}\right)_{(m+\epsilon_0)}$ be an orthonormal basis for a Hilbert space H. Let $f_1 \in H$ with $\|f_1\| = 1$, then $(f_1, e_2, e_3, \ldots)$ is a Riesz basis for H if and only if $\langle f_1, e_1 \rangle \neq 0$. Verifying that the map T, given by $e_k \longmapsto e_k$ for $k > 1$ and $e_1 \longmapsto f_1$, is a continuous bijection is routine, so T is an isomorphism via the Open Mapping theorem. In the language of Theorem 5.1, $(f_1, e_2, e_3, \ldots)$ is a Riesz basis for $L_2[-\pi,\pi]$ if and only if $0 \neq sinc(\pi t_{1,1}) \cdot \cdots \cdot sinc(\pi t_{1,d})$, that is, if and only if
$t_1 \in (\mathbb{C}\backslash\{\pm 1, \pm 2, \ldots\})^d$.

VI. Generalizations of Kadec's 1/4 theorem

Corollary 5.4 yields the following generalization of Kadec's theorem in d dimensions (see [13]) .

Corollary 6.1 . Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$ and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{C}^d$ such that

$$\sup_{k\in\mathbb{Z}} \|(m+\epsilon_0)_k - t_k\|_\infty = L < \frac{\ln(2)}{\pi d}. \tag{22}$$

Then the sequence $(f_k)_{k\in\mathbb{N}}$ defined by $f_k(x) = \frac{1}{(2\pi)^{d/2}} e^{i\langle x, t_k\rangle}$ is a Riesz basis for $L_2[-\pi,\pi]^d$.

The proof is immediate. Note that (20) implies that the map T given in Corollary 5.4 has norm less than 1. We conclude that the map $(I - T)e_k = f_k$ is invertible by considering its Neumann series.

The proof of Corollary 5.4 and Corollary 6.1 are straightforward generalizations of the univariate result proved by Duffin and Eachus [8]. Kadec improved the value of the constant in the inequality (22) (for $d = 1$) from $\frac{\ln(2)}{\pi}$ to the optimal value of 1/4; this is his celebrated "1/4 theorem" [10].Kadec's method of proof is to expand $e^{i\delta x}$ with respect to the orthogonal basis

$$\left\{1, cos((m+\epsilon_0)x), sin\left((m+\epsilon_0) - \frac{1}{2}\right)x\right\}_{(m+\epsilon_0)\in\mathbb{N}}$$

for $L_2[-\pi,\pi]$, and use this expansion to estimate the norm of T.

In the proof of Corollary 5.4 and Corollary 6.1 we simply used a Taylor series (see [13]). Unlike the estimates in Kadec's theorem, the estimate in (20) can be used for any sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{C}^d$such that $sup_{k\in\mathbb{Z}}\|(m+\epsilon_0)_k - t_k\|_\infty = L < \infty$, not only those for which the exponentials $(e^{it_{(m+\epsilon_0)}x})_{(m+\epsilon_0)}$ form a Riesz basis. An impressive generalization of Kadec's 1/4 theorem when $d = 1$ is Avdonin's "1/4 in the mean" theorem [1].Sun and Zhou (see [12] second half of Theorem 1.3) refined Kadec's argument to obtain a partial generalization of his result in higher dimensions:

Theorem 6.2 . Let $(a_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}^d} \subset \mathbb{C}^d$ such that $0 < L < \frac{1}{4}$,

$$D_d(L) = \left(1 - cos\pi L + sin\pi L + \frac{sin\pi L}{\pi L}\right)^d - \left(\frac{sin\pi L}{\pi L}\right)^d ,$$

and

$\left\|a_{(m+\epsilon_0)} - (m+\epsilon_0)\right\|_\infty \leq L, (m+\epsilon_0) \in \mathbb{Z}^d$. If $D_d(L) < 1$, then $\left(\frac{1}{(2\pi)^d} e^{i\langle a_{(m+\epsilon_0)},(\cdot)\rangle}\right)$ is a Riesz basis for $L_2[-\pi,\pi]^d$ with frame bounds $(1 - D_d(L))^2$ and $(1 + D_d(L))^2$ .In the one-dimensional case, Kadec's theorem is recovered exactly from Theorem 6.2. When $d > 1$, the value x_d satisfying $0 < x_d < \frac{1}{4}$ and $D_d(x_d) = 1$ is an upper bound for any value of L satisfying $0 < L < \frac{1}{4}$ and $D_d(L) < 1$. The value of x_d is not readily apparent, whereas the constant in Corollary 6.1 is $\frac{\ln 2}{x_d}$. A relationship between this number and x_d is given in the following theorem (whose proof is omitted).

Theorem 6.3 . Let x_d be the unique number satisfying $0 < x_d < \frac{1}{4}$ and $D_d(x_d) = 1$. Then

$$\lim_{d\to\infty} \frac{x_d - \frac{\ln 2}{\pi d}}{\frac{(\ln 2)^2}{12\pi d^2}} = 1 .$$

Thus, for sufficiently large d, Theorem 6.2 and Corollary 6.1 are essentially the same.

7. A method of approximation of biorthogonal functions and a recovery of a theorem of Levinson

In this section we apply (see [13]) the techniques developed previously to approximate the biorthogonal functions to Riesz bases $\left(\frac{1}{\sqrt{2\pi}} e^{it_{(m+\epsilon_0)}(\cdot)}\right)$ for which the synthesis operator is small perturbation of the identity.

Definition 7.1. A Kadec sequence is a sequence $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}}$ of real numbers satisfying $\sup_{(m+\epsilon_0)\in\mathbb{Z}} \left|t_{(m+\epsilon_0)} - (m+\epsilon_0)\right| = D < \frac{1}{4}$.

Theorem 7.2 . Let $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence (with $t_{(m+\epsilon_0)} \neq 0$ for $(m+\epsilon_0) \neq 0$) such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)} = \left(\frac{1}{\sqrt{2\pi}} e^{it_{(m+\epsilon_0)}(\cdot)}\right)_{(m+\epsilon_0)}$ is a Riesz basis for $L_2[-\pi,\pi]$, and let $(e_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ be the standard exponential orthonormal basis for $L_2[-\pi,\pi]$. If the map L given by $Le_{(m+\epsilon_0)} = f_{(m+\epsilon_0)}$ satisfies the estimate $I - L < 1$, then the biorthogonals $G_{(m+\epsilon_0)}$ of $\frac{1}{\sqrt{2\pi}}\mathcal{F}(f_{(m+\epsilon_0)})(\cdot) = sinc(\pi(\cdot - t_{(m+\epsilon_0)}))$ in $PW_{[-\pi,\pi]}$ are

$$G_{(m+\epsilon_0)}(t) = \frac{H(t)}{(t - t_{(m+\epsilon_0)})H'(t_{(m+\epsilon_0)})}, \qquad (m+\epsilon_0) \in \mathbb{Z} , \tag{23}$$

where

$$H(t) = (t - t_0) \prod_{(m+\epsilon_0)=1}^{\infty} \left(1 - \frac{t}{t_{(m+\epsilon_0)}}\right)\left(1 - \frac{t}{t_{-(m+\epsilon_0)}}\right) . \tag{24}$$

Definition 7.3 . Let $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)} = \left(\frac{1}{\sqrt{2\pi}} e^{it_{(m+\epsilon_0)}(\cdot)}\right)_{(m+\epsilon_0)}$ is a Riesz basis for $L_2[-\pi,\pi]$. If $l \geq 0$, the$l$-truncated sequence $(t_{l,(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}}$ is defined by $t_{l,(m+\epsilon_0)} = t_{(m+\epsilon_0)}$ if $|(m+\epsilon_0)| \leq l$ and $t_{l,(m+\epsilon_0)} = (m+\epsilon_0)$ otherwise. Define $f_{l,(m+\epsilon_0)} = \frac{1}{\sqrt{2\pi}} e^{it_{l,(m+\epsilon_0)}(\cdot)}$ for $(m+\epsilon_0) \in \mathbb{Z}, l \geq 0$.

Let $P_l : L_2[-\pi,\pi] \to L_2[-\pi,\pi]$ be the orthogonal projection onto span$\{e_{-l},\dots,e_l\}$.

Proposition 7.4 . Let $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ (defined above) is a Riesz basis for $L_2[-\pi,\pi]$. If $(e_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ is the standard exponential orthonormal basis for $L_2[-\pi,\pi]$ and the map L (defined above) satisfies the estimate $\|I - L\| = \delta < 1$, then the following are true:

(i) For $l \geq 0$, the sequence $(f_{l,(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a Riesz basis for$L_2[-\pi,\pi]$.

(ii) For $l \geq 0$, the map L_l defined by $L_l e_{(m+\epsilon_0)} = f_{l,(m+\epsilon_0)}$ satisfies $\|L_l^{-1}\| \leq \frac{1}{1-\delta}$.

Proof . If $(c_{(m+\epsilon_0)})_{(m+\epsilon_0)} \in \ell^2(\mathbb{Z})$, then

$$(I - L_l)\left(\sum_{(m+\epsilon_0)} c_{(m+\epsilon_0)} e_{(m+\epsilon_0)}\right) = \sum_{(m+\epsilon_0)} c_{(m+\epsilon_0)}\left(e_{(m+\epsilon_0)} - L_l e_{(m+\epsilon_0)}\right) = \sum_{|(m+\epsilon_0)|\leq l} \left(e_{(m+\epsilon_0)} - f_{(m+\epsilon_0)}\right)$$

$$= (I - L)P_l\left(\sum_{(m+\epsilon_0)} c_{(m+\epsilon_0)} e_{(m+\epsilon_0)}\right),$$

so that

$$(I - L_l) = (I - L)P_l \; . \tag{25}$$

From this, $\|I - L_l\| \leq \delta$, which implies (i) and (ii).

Define the biorthogonal functions of $(f_{l,(m+\epsilon_0)})_{(m+\epsilon_0)}$ to be $(f^*_{l,(m+\epsilon_0)})_{(m+\epsilon_0)}$. Passing to the Fourier transform, we have $\frac{1}{\sqrt{2\pi}}\mathcal{F}(f_{l,(m+\epsilon_0)})(t) = sinc(\pi(t - t_{l,(m+\epsilon_0)}))$ and $G_{l,(m+\epsilon_0)}(t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f^*_{l,(m+\epsilon_0)})(t)$. Define the biorthogonal functions of $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ similarly.

Lemma 7.5 . If $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)} \subset \mathbb{C}$ satisfies the hypotheses of Proposition 7.4, then

$$\lim_{l\to\infty} G_{l,(m+\epsilon_0)} = G_{(m+\epsilon_0)} \quad \text{in} \quad PW_{[-\pi,\pi]} .$$

Proof . Note that $\delta_{(m+\epsilon_0)m} = \langle f_{l,(m+\epsilon_0)}, f^*_{l,m}\rangle = \langle L_l e_{(m+\epsilon_0)}, f^*_{l,m}\rangle = \langle e_{(m+\epsilon_0)}, L^*_l f^*_{l,m}\rangle$ so that for all $m, f^*_{l,m} = (L^*_l)^{-1} e_m$. Similarly, $f^*_m = (L^*)^{-1} e_m$. We have $f^*_{l,m} - f^*_m = ((L^*_l)^{-1} (L^*)^{-1}) e_m = (L^*_l)^{-1}(L^* - L^*_l)(L^*)^{-1} e_m$. Now (84) implies $L - L_l = (I - P_l)(L - I)$, so that $f^*_{l,m} - f^*_m = (L^*_l)^{-1}(L^* - I)(I - P_l)(L^*)^{-1} e_m$. Applying Proposition 7.4 yields $\|f^*_{l,m} - f^*_m\| \leq \frac{1}{1-\delta}\|(L^* - I)(I - P_l)(L^*)^{-1} e_m\|$, which for fixed m goes to 0 as $l \to \infty$. We conclude $\lim_{l\to\infty} f^*_{l,m} = f^*_m$, which, upon passing to the Fourier transform, yields $\lim_{l\to\infty} G_{l,m} = G_m$.

Proof of Theorem 7.2. We see that $\delta_{(m+\epsilon_0)m} = \langle G_{l,m}, S_{l,(m+\epsilon_0)}\rangle$, where $S_{l,(m+\epsilon_0)}(t) = sinc(\pi(t - t_{(m+\epsilon_0)}))$ when $|(m+\epsilon_0)| \leq l$ and $S_{l,(m+\epsilon_0)}(t) = sinc(\pi(t - (m+\epsilon_0)))$ when $|m| > l$. Without loss of generality, let $|m| < l$. (1) implies that $G_{l,m}(k) = 0$ when $|k| > l$. By the WKS theorem we have

$$G_{l,m}(t) = \sum_{k=-l}^{k=l} G_{l,m}(k) sinc(\pi(t-k)) = \left(\sum_{k=-l}^{k=l} \frac{(-1)^{k-1} t G_{l,m}(k)}{k - t}\right) sinc(\pi t) = \frac{w_l(t)}{\prod_{k=1}^{l}(k-t)(-k-t)} sinc(\pi t),$$

where w_l is a polynomial of degree at most $2l$. Noting that

$$sinc(\pi t) = \prod_{k=1}^{\infty}\left(1 - \frac{t^2}{k^2}\right) \text{ and } \prod_{k=1}^{l}(k-t)(-k-t) = (-1)^l (l!)^2 \prod_{k=1}^{l}\left(1 - \frac{t^2}{k^2}\right),$$

we have

$$G_{l,m}(t) = \frac{(-1)^l w_l(t)}{(l!)^2} \prod_{k=l+1}^{\infty}\left(1 - \frac{t^2}{k^2}\right).$$

Again by (1), $\delta_{(m+\epsilon_0)m} = G_{l,m}(t_{(m+\epsilon_0)})$ when $|(m+\epsilon_0)| \leq l$ so that

$$\delta_{(m+\epsilon_0)m} = \frac{(-1)^l}{(l!)^2} w_l(t_{(m+\epsilon_0)}) \prod_{k=l+1}^{\infty}\left(1 - \frac{t^2_{(m+\epsilon_0)}}{k^2}\right).$$

This determines the zeroes of w_l. We deduce that

$$w_l(t) = \frac{c_l \prod_{k=1}^{k=l}(t - t_k)(t - t_{-k})}{t - t_m}$$

for some constant c_l . Absorbing constants, we have $G_{l,m}(t) = \frac{c_l H_l(t)}{t - t_m}$, where

$$H_l(t) = (t - t_0)\prod_{k=1}^{l}\left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right)\prod_{l+1}^{\infty}\left(1 - \frac{t^2}{k^2}\right).$$

Now $0 = H_l(t_m)$, so $G_{l,m}(t) = c_l \frac{H_l(t) - H_l(t_m)}{t - t_m}$. Taking limits, $c_l = \frac{1}{(H_l)'(t_m)}$. This yields $G_{l,m}(t) = \frac{H_l(t)}{(t - t_m)H'_l(t_m)}$. Define

$$H(t) = (t - t_0)\prod_{k=1}^{\infty}\left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right).$$

Basic complex analysis shows that H is entire, and $H_l \to H$ and $H'_l \to H'$ uniformly on compact subsets of C. Furthermore, $H'(t_k) \neq 0$ for all k, since each t_k is a zero of H of multiplicity one. Together we have

$$\lim_{l\to\infty} G_{l,m}(t) = \frac{H(t)}{(t - t_m)H'(t_m)}, t \in \mathbb{C}.$$

By the foregoing lemma, $G_{l,m} \to G_m$. Observing that convergence in $PW_{[-\pi,\pi]}$ implies pointwise convergence yields the desired result.Levinson proved a version of Theorem 7.2 in the case where $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}}$ is a Kadec sequence. His original proof is found in [11, pp. 47–67]. We recall that if $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a Riesz basis arising from a Kadec sequence, then the synthesis operator L satisfies $\|I - L\| < 1$. Levinson's theorem is then recovered from Theorem 7.2.

Now we show the following Corollary

Corollary 7.6. Let $(2t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence and $\left(f^2_{2(m+\epsilon_0)}\right)_{(m+\epsilon_0)}$ is a Riesz basis for $L_2[-\pi,\pi]$. If $(e_{2(m+\epsilon_0)})_{(m+\epsilon_0)}$ is the standard exponential orthonormal basis for $L_2[-\pi,\pi]$ and the map L^2 satisfies the estimate $\|I - L^2\| = \delta < 1$, then the following are hold:

(i) For $l \geq 0$, the sequence $\left(f^2_{2l,2(m+\epsilon_0)}\right)_{(m+\epsilon_0)}$ is a Riesz basis for $L_2[-\pi,\pi]$.

(ii) For $l \geq 0$, the map L^2_{2l} defined by $L^2_{2l}e_{2(m+\epsilon_0)} = f^2_{2l,2(m+\epsilon_0)}$ satisfies $\|L^{-2}_{2l}\| \leq \frac{1}{(1-\delta)^2}$.

Proof . For $(c_{2(m+\epsilon_0)})_{(m+\epsilon_0)} \in \ell^2(\mathbb{Z})$, we have

$$(I - L^2_{2l})\left(\sum_{(m+\epsilon_0)} c_{2(m+\epsilon_0)}e_{2(m+\epsilon_0)}\right) = \sum_{(m+\epsilon_0)} c_{2(m+\epsilon_0)}\left(e_{2(m+\epsilon_0)} - L^2_{2l}e_{2(m+\epsilon_0)}\right)$$

$$= \sum_{|(m+\epsilon_0)|\leq l}\left(e_{2(m+\epsilon_0)} - f^2_{2(m+\epsilon_0)}\right) = (I - L^2)P_{2l}\left(\sum_{(m+\epsilon_0)} c_{2(m+\epsilon_0)}e_{2(m+\epsilon_0)}\right),$$

so that $(I - L^2_{2l}) = (I - L^2)P_{2l}$.

Hence $\|I - L^2_{2l}\| \leq \delta$, which gives (i) and (ii). Hence from Definition 2.4 we can show that

$$A\sum_j \|f_j\|^2 \leq \sum_{(m+\epsilon_0)}\sum_j \left|\left\langle f_j, \left(f^2_{2l,2(m+\epsilon_0)}\frac{Le_{(m+\epsilon_0)}}{L^2_{2l}e_{2(m+\epsilon_0)}}\right)\right\rangle\right|^2 \leq (A+\epsilon_1)\sum_j \|f_j\|^2, \text{for every } f_j \in H, \epsilon_1 > 0 .$$

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