International Journal Of Research In Engineering And Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
Www.Ijres.Org Volume 5 Issue 11 ǁ November. 2017 ǁ PP. 09-14

Stability of Hopfield Lineal Network in Continuous Time

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ABSTRACT: In this work a sufficient condition is given to guarantee the stability of the Hopfiel linear network in continuous time (LHN), since the LHN is used to solve the WiennerHopf equation. Finally, the postulated is verified by the simulation in Matlab of the transient response of an LHN of order 4

Keywords: Continuous-time linear Hopfield network, linear system, stability, symmetric matrix, WiennerHopf.

Date of Submission: 30-10-2017 Date of acceptance: 09-11-2017

I. INTRODUCTION

Fig.1 shows the structure of a Wienner filter.

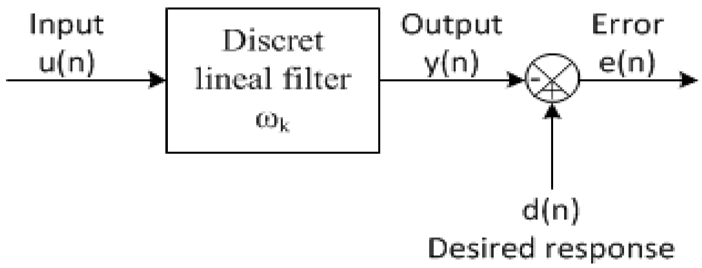

Figure 1.Wienner Filter.

The objective of the system is to determine the optimal set of weights $\{\omega_k\}$ such that they minimize the difference in the mean squared error between the output y(n) and a desired response d(n). The output of the filter y(n) is given in (1):

$$y(n) = \sum_{k=0}^{\infty} \omega_k u(n-k) \quad (1)$$

And the error in (2):

$$e(n) = d(n) - y(n) \quad (2)$$

As a measure of filter performance, the mean square error in (3) is defined by:

$$J = \frac{1}{2} E[e^2(n)] \quad (3)$$

Where E[·] represents expected value. The mean square error has its optimal value J_{min}, (see in [1]) when the WiennerHopf equation given by (4) is satisfied:

$$\sum_{j=0}^{\infty} \omega_{ok} r_{xx}(j-k) = r_{xd}(k), \text{for } k = 1,2,\ldots \quad (4)$$

whereω_{ok} is the k-th optimum weight, r_{xx}(j-k) in (5):

$$r_{xx}(j-k) = E[u(n-k)u(n-j)] \quad (5)$$

is the auto-correlation function of the input, and r_{xd}(k) in (6):

$$r_{xd}(k) = E[d(n)u(n-k)] \quad (6)$$

is the cross-correlation function between the desired input and output. Note that (5) is symmetric, that is, r_{xx}(j–k)=r_{xx}(k–j). Defining,

$$\mathbf{u(n)} = [u(n), u(n-1), u(n-2), \ldots]^T$$

the auto-correlation matrix is obtained in (7):

$$\mathbf{R} = \mathrm{E}[\mathbf{u}(\mathbf{n})\mathbf{u}^{\mathbf{T}}(\mathbf{n})] = \begin{pmatrix} r_{xx}(0) & r_{xx}(1) & \cdots \\ r_{xx}(1) & r_{xx}(0) & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \quad (7)$$

and the cross-correlation vector in (8):

$$\mathbf{p} = \mathrm{E}[\mathbf{u}(\mathbf{n})\mathrm{d}(\mathrm{n}))] = [\mathrm{u}(\mathrm{n})\mathrm{d}(\mathrm{n}) \quad \mathrm{u}(\mathrm{n}-1)\mathrm{d}(\mathrm{n}) \quad \cdots]^{\mathrm{T}} \quad (8)$$

so that, the Wiener Hopf equation in matrix form is given in (9):

$$\mathbf{R}\boldsymbol{\omega}_{\mathbf{o}} = \mathbf{p} \quad (9)$$

The problem of finding the optimal weights in the Wienner filter requires infinite information, so that it is replaced by a finite filter and instead of solving (9), adaptive filtering is used. The basic structure of an adaptive filter is shown in Fig.2 The main characteristic of adaptive systems is their adaptability, that is, instead of being built by specification, they use the instantaneous error to fix their weights through systematic procedures called rules or learning algorithms (for a complete review of the popular algorithms see [2]), whose object is to minimize the error in the mean quadratic sense.

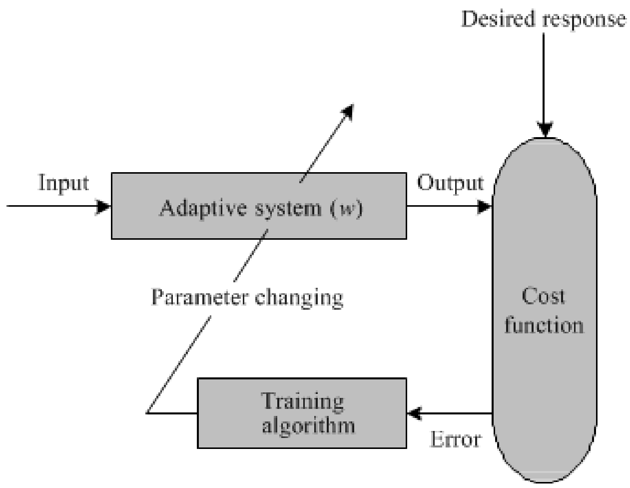

Figure 2.Structure of an adaptive filter.

Continuous-time LHN is a good alternative to solve by estimators of the autocorrelation matrix R and the cross-correlation vector p the Wiener Hopf equation, without the need to resort to the matrices inversion process, since its response in steady state is similar to the solution in (9). However, for this to happen the stability of the LHN must be guaranteed. The LHN is not a learning algorithm, so it must receive the input of the estimators of **R**, **p**to obtain the optimal parameters $\boldsymbol{\omega}_{\mathbf{0}}$. In this way the LHN provides the best solution to the problem. In the present work a sufficient condition is provided so that the LHN of any order is stable. This is of paramount importance, since it guarantees its stability in a general way.

The work is organized as follows: Section 2 sets forth the definitions and propositions of matrices stability theory necessary for the development of the work. In section 3 the Hopfield neural network in continuous time (HNN) and its variant, the LHN, is introduced. Likewise, a theorem is postulated and demonstrated in which it establishes the sufficient condition for the LHN to be stable. In this same section, the system response is obtained in steady state when the input signals are constant. In section 4 the condition in the previous section is verified by a simulation in Matlab. Finally, in section 5 the conclusions of the work, its limitations and future work are given.

II. PRELIMINARY

From the reference [3] most of the definitions and propositions that were enunciated in this section and that are useful to establish the condition of stability of the LHN were taken.

Definition 2.1. Let A be a matrix of size n, with entries in, A is a Hermitian matrix if A=A*, where A* is the transposed conjugate of A. If A has only entries in, it will be said that A is symmetric and $A^*=A^T$.

Definition 2.2. Let A be a Hermitian matrix of size n, then:

• A is positive definite, if $x^*Ax > 0$for all $x \in \mathbb{C}^n$

• A is defined negative, if$x^*Ax < 0$for all $x \in \mathbb{C}^n$

Proposition 2.3. Let A be a real matrix of size n, if A is symmetric, then all its eigenvalues are real.

Demonstration

Let λ be an eigenvalue of A, then it is true that for a vector x≠0 in $\mathbb{C}^n$, Ax=λx Then,

$$x^*Ax = (A^Tx)^*x$$
$$x^*(Ax) = (Ax)^*x$$
$$x^*(\lambda x) = (\lambda x)^*x$$
$$\lambda(x^*x) = \bar{\lambda}(x^*x)$$
$$(\lambda - \bar{\lambda})(x^*x) = 0$$

Concluding therefore, $\lambda \in \mathbb{R}$.

Proposition 2.4. Let A be a real and symmetric matrix of size n, then A is definite negative if, and only if, all its eigenvalues are negative.

Demonstration

First, suppose that A is defined as negative. Let x be an eigenvector associated with the eigenvalue λ. So,

$$x^TAx = x^T\lambda x = \lambda x^Tx = \lambda\|x\|^2 < 0$$

Since A is symmetric all the eigenvalues are real, concluding that λ<0.

Now suppose that all eigenvalues are negative. Since A is symmetric all its eigenvalues are real, in addition there is an orthonormal basis formed by all the eigenvectors of A, namely, $\{x_1,x_2,\ldots,x_n\}$, such that any x≠0 con $x\in\mathbb{R}^n$ $x=a_1x_1+a_2x_2+\ldots+a_nx_n$. So,

$$x^TAx = \lambda\|x\|^2 = \lambda(a_1^2\|x_1\|^2 + a_2^2\|x_2\|^2 + \ldots + a_n^2\|x_n\|^2) < 0$$

Definition 2.5. For $z\in\mathbb{R}$ it will be said that, with $P(z) = z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$, with $a_i\in\mathbb{R}$ it is a polynomial of Hurwitz, if all its roots have real negative part.

Definition 2.6. The characteristic polynomial of a matrix A of dimension n is given by:

$$p(\lambda) = det(\lambda I - A) = \lambda^n + C_1\lambda^{n-1} + \cdots + C_{n-1}\lambda + C_n$$

Where all the $C_i$represent the sum of all the minor principles signed in the order of i (see in [4])

Proposition 2.7.A real, symmetric matrix of size n is defined negative, if and only if, its characteristic polynomial is a Hurwitz polynomial.

Demonstration

If A is defined negative all the roots of its characteristic polynomial (eigenvalues) are negative, and therefore, it is concluded that it is a polynomial of Hurwitz. On the other hand, if its characteristic polynomial is a Hurwitz polynomial all its roots (eigenvalues) have real negative part, then as A is symmetric its eigenvalues are real and negative, concluding so that A is defined negative.

1. Continuous-time Hopfield network

The Hopfield network of order n in continuous time (see in [5]) bases its realization on the circuit shown in Fig. 3 (see [6]) and described in (10):

$$C\frac{dx_i}{dt} = -\frac{1}{R}x_i + \sum_{j=1}^{n} w_{ij}f(x_j) + b_i \qquad (10)$$

For i=1,...,n, where x_i (voltage signal) represents the activity of the ith-neuron, w_{ij} (gain of the transconductance amplifier), the synaptic weight corresponding to the synaptic connection between the jth-neuron and the neuron i-th is symmetrical, b_i (current signal) is an external input, R (electrical resistance), and C (electric capacitance) are positive constants and f(·) (activation function) is a non-linear function.

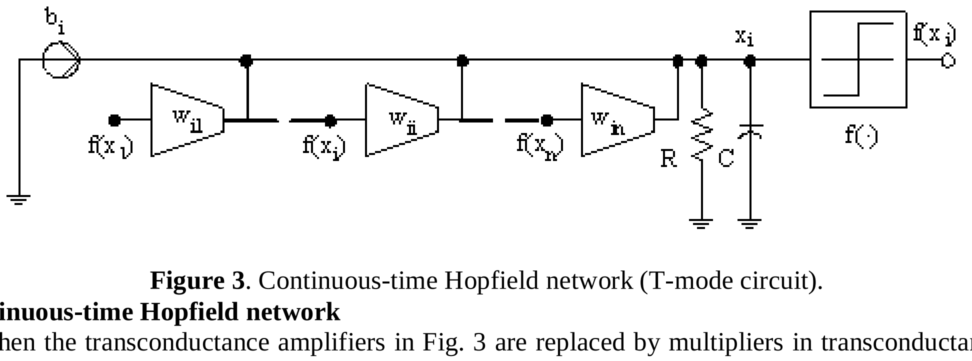

Figure 3. Continuous-time Hopfield network (T-mode circuit).

1.1. Continuous-time Hopfield network

Then the transconductance amplifiers in Fig. 3 are replaced by multipliers in transconductance mode, such that $w_{ij}=g_mv_{ij}$. In this case, g_m represents the gain of the multiplier and v_{ij} is an external input with voltage dimensions. Also, the activation function is replaced by the linear function f(x)=x, or what is the same, a feedback loop is introduced at the output of each neuron as shown in Fig. 4

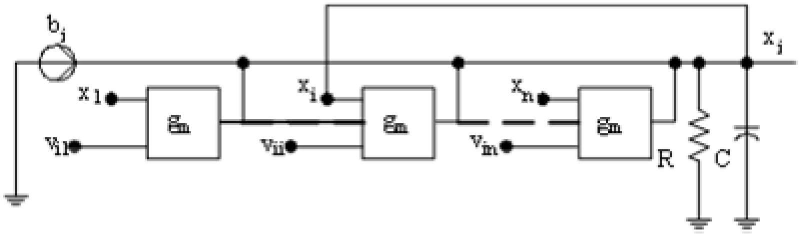

Figure 4.Continuous-time linear Hopfield network.

The model in Fig. 4 represents the linear Hopfield network in continuous time (LHN) and is described by the system of linear equations in (11):

$$C\frac{dx_i}{dt} = -\frac{1}{R}x_i + g_m\sum_{j=1}^{n} v_{ij}x_j + b_i \qquad (11)$$

For i=1, ..., n, written in matrix form is given by (12):

$$\dot{\boldsymbol{x}} = -\omega_0(\boldsymbol{I} - g_m R\boldsymbol{V})\boldsymbol{x} + \omega_0(R\boldsymbol{b}) \qquad (12)$$

Where ω_0=1/RC is a positive constant having angular frequency dimensions (rad/s); **I** is the identity matrix of size n; **V** is a symmetric matrix of size n×1 and **b** is an array of size n×1 that contains the entries to the system and x an array of size n×1 that represents the output of the system.

1.2. Stability

This section gives a sufficient condition for the LHN to be stable. We begin by rewriting the system in (12) by the equation of state in (13):

$$\dot{\boldsymbol{x}} = \boldsymbol{Bx} + \boldsymbol{Cb} \qquad (13)$$

The equation given in (13) represents a linear system, where B=–ω_0A, is the coefficient matrix of the homogeneous system, of size n×n with A=I–g_mRV, C is a diagonal matrix of size n×n with all its diagonal elements given by ω_0R and **b** is the input signal matrix of size n×1.

Definition 3.1. A linear system as described in (13) is stable if the characteristic polynomial of matrix **B** is a Hurwitz polynomial.

Definition 3.2. An array A of size n is strictly diagonal by row if the condition in (14) is satisfied:

$$|a_{ii}| > \sum_{j\neq i}|a_{ij}| \qquad (14)$$

For i = 1, ..., n. It is said that A is strictly diagonal by column, if A^T is strictly diagonal by row.

Proposition 3.3. Let A be a real matrix, symmetric and strictly diagonal dominant. If $a_{ii}>0$ for i=1, ..., n, then A is defines defined positive.

Demonstration

It should be shown that $x^TAx>0$ for each x≠0, with x on ℝ. First, since A is strictly diagonal dominant and $a_{ii}>0$, (15) follows from (14):

$$a_{ii} - \sum_{j\neq i}|a_{ij}| > 0 \qquad (15)$$

For i=1, ..., n. On the other hand, x^TAx is developed in (16):

$$x^TAx = \sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i=1}^{n}\sum_{j\neq i} a_{ij}x_ix_j = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i=1}^{n}\sum_{j=i+1}^{n}(a_{ij} + a_{ji})x_ix_j \qquad (16)$$

Completing perfect square trinomial in the second sum term,

$$(a_{ij} + a_{ji})x_ix_j = \frac{|a_{ij} + a_{ji}|}{2}(x_i \pm x_j)^2 - \left(\frac{|a_{ij} + a_{ji}|}{2}x_i^2 + \frac{|a_{ij} + a_{ji}|}{2}x_j^2\right)$$

(17) is obtained:

$$x^TAx = \sum_{i=1}^{n}\left(a_{ii}x_i^2 - x_i^2\sum_{j\neq i}^{n}\frac{|a_{ij} + a_{ji}|}{2}\right) + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{|a_{ij} + a_{ji}|}{2}(x_i \pm x_j)^2 \qquad (17)$$

Considering that A is symmetric, $a_{ij}=a_{ji}$, (17) can be rewritten as,

$$x^T A x = \sum_{i=1}^{n} x_i^2 \left(a_{ii} - \sum_{j \neq i}^{n} |a_{ij}| \right) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{|a_{ij} + a_{ji}|}{2} (x_i \pm x_j)^2$$

Concluding from (15) that $x^T Ax>0$.

Theorem 3.4.If $|v_{ij}| < \frac{1}{n g_m R}$ for i,j=1, ..., n where n is the order of the LHN, then the LHN of order n is stable.

III. DEMONSTRATION

Given that the matrix **B** is symmetric, according to definition 3.1 and proposition 2.7 it suffices to prove that it is defined negative or that **A** is defined positive. For this, in the light of proposition 3.3, it must be fulfilled that $a_{ii}> 0$ and that **A** is strictly diagonal dominant.

First, it is observed that, the elements in the main diagonal of the matrix **A** are given by $a_{ii} = 1 - g_m R v_{ii}$ for i,j=1,….n. Using the hypothesis it follows that the $a_{ii} = 1 - g_m R v_{ii}$ are contained in the open interval $\left(\frac{n-1}{n}, \frac{n+1}{n}\right)$, so that $a_{ii}>0$ for i=1,...n Now consider the elements outside the diagonal of **A**, $a_{ij} = -g_m R v_{ij}$ for j≠i using the hypothesis again, it follows that $|a_{ij}| < \frac{1}{n}$ it follows that $\sum_{j\neq i} |a_{ij}| < \frac{n-1}{n}$ for i = 1, ..., n, satisfying (14) and concluding that **A** is defined positive.

1.3. Steady-state response

The transient response of the system in (13) is given by (18) (see [7]):

$$\boldsymbol{x}(t) = e^{\boldsymbol{B}t}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{B}(t-\tau)} \boldsymbol{Cb}(\tau) d\tau \qquad (18)$$

Where, $\mathbf{x}(t_0)$ is a start condition for $t_0 \geq 0$ and $e^{\mathbf{B}t}$ is a matrix containing terms of the form, $t^{(\alpha_i - 1)} e^{\lambda_i t}$ fori=1,…k with α_i multiplicity of each eigenvalue λ. On the other hand, if t=0, $e^{\mathbf{B}(0)}=\mathbf{I}$. Ifb(t)=b=cte. (18) has the solution in (19):

$$\boldsymbol{x}(t) = e^{\boldsymbol{B}t}\boldsymbol{x}(t_0) - \boldsymbol{B}^{-1}\left(\boldsymbol{I} - e^{\boldsymbol{B}(t-t_0)}\right)\boldsymbol{Cb} \qquad (19)$$

If the system is stable, all its eigenvalues are negative, followed by $\lim_{t\to\infty} e^{\mathbf{B}t} = 0$ therefore, the steady-state response of the system is as in (20),

$$\boldsymbol{x}_{ss} = -\boldsymbol{B}^{-1}\boldsymbol{Cb} \qquad (20)$$

Substituting $\mathbf{B}=-\omega_0(\mathbf{I}-g_m R\mathbf{V})$, $\mathbf{C}=\omega_0 R\mathbf{I}$ in (20), we obtain the expression in (21):

$$\boldsymbol{x}_{ss} = (\boldsymbol{I} - g_m R\boldsymbol{V})^{-1}(R\boldsymbol{b}) \qquad (21)$$

The speed, with which the system reaches the steady state, depends on the dominant eigenvalue (the one closest to 0).

2. Simulation and results

In this section a simulation is performed in Matlab to verify Teo.3.4 and (21), the procedure used is listed below.

Step 1: The constants were defined g_m=25μA/V², R=1kΩ y C=100μF n=4 (order of the LHN) and b_{max}=5mA (maximum allowable current at the network input). Thus, according to Teo.3.4, in order for the RNL to be stable, it must be satisfied (22):

$$|v_{ij}| < 10V fori, j = 1,2,3,4 \qquad (22)$$

Step 2: The matrix C of the state equation in (13), is obtained by multiplying $\omega_0 R\mathbf{I}$, dondeω$_0$=1/(RC).

Step 3: The matrix $\mathbf{B}=-\omega_0(\mathbf{I}$-gmR$\mathbf{V})$ of the homogeneous state equation in (13) was obtained by generating the real and symmetric voltage matrix **V** at random, where each element of v_{ij} for i,j=1,2,3,4, is a discrete random variable with uniform distribution in the set {-9,-8,…,8,9}V, which guarantees the fulfillment of (22). For this, we used the **randi** instruction of Matlab.

Step 4: Each element of matrix **b** in (13) was obtained similarly to v_{ij} with the randi instruction, but in the discrete set {1,…,b_{max}}.

Step 5: The dominant eigenvalue (the maximum) of matrix **B** is obtained and the parameter $\tau=-1/\lambda_{max}$. is defined.

Step 6: The output x of (13) is obtained in a transient regime with the instruction **lsim** of Matlab, giving a time interval of duration 5τ, in order that the system reaches its steady state $\mathbf{x}_{ss}$ given in (21).

Step 7: The direct calculation of $\mathbf{x}_{ss}$ is performed in (21) and plotted together with the transient response that was obtained in step 6.

Steps 3 to 7 were performed 3 times in sequence with different matrices **B** and **b**. The initial state of each stage $\mathbf{x}_0$ corresponds to the final state of the previous stage. The initial state of the first stage was set to **0**. The results of the simulation are shown in Fig. 5.

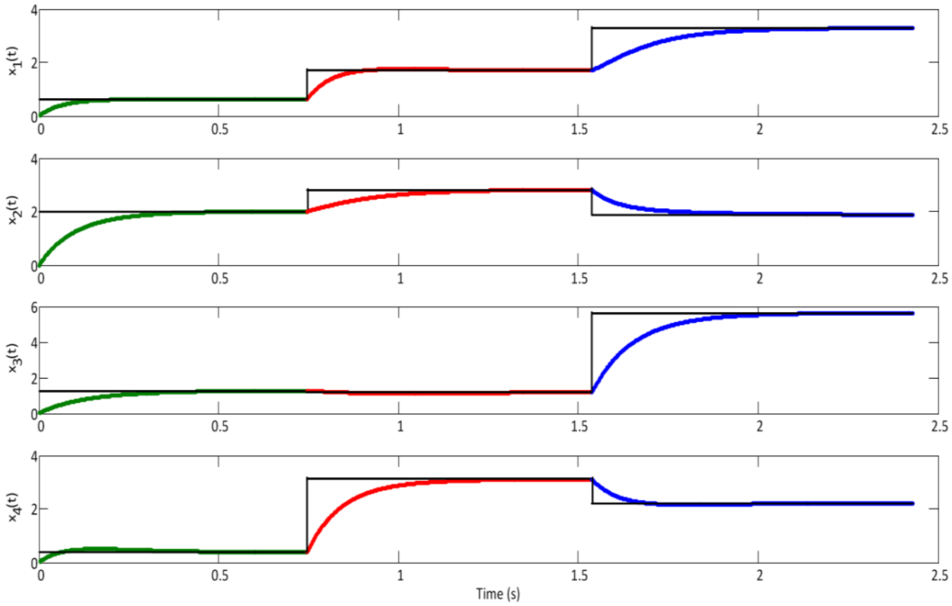

Figure 5.Transient response of an LHN of order 4.

IV. CONCLUSION

In the present work a sufficient condition is given for the LHN to be stable under specific constraints of the input voltage variables, v_{ij}, for i,j=1,1,…,n. Ensuring the stability of the LHN is of paramount importance to solve the Wiener Hopf equation without resorting to matrix inversion. On the other hand, the region of frequency operation of the network depends on the dominant eigenvalue of the matrix of the homogeneous equation **B**, ignoring the dependence of said eigenvalue with the parameters of the network is a limiting factor for its operation when the signal of input **b**=**b**(t) is time dependent. Therefore, it is necessary in the future to perform the frequency analysis of the LHN.

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*Silvestre Ascención García Sánchez. "Stability of Hopfield Lineal Network in Continuous Time." International Journal Of Research In Engineering And Science (IJRES), vol. 05, no. 11, 2017, pp. 09–14.