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A Short Report on Different Wavelets and Their Structures.

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Abstract: This article consists of basics of wavelet analysis required for understanding of and use of wavelet theory. In this article we briefly discuss about HAAR wavelet transform their space and structures.

Keywords: *HAAR wavelet transform, Multiresolution analysis (MRA), Daubechies wavelet transforms, Biorthogonal wavelets.*
Mathematical Classification: 42C40

I. Introduction to wavelets [1,2,6,7]

Wavelet analysis is used to decomposes sounds and images into component waves of varying durations, called wavelets which are localised vibrations of a sound signal or localized detail in an image. This analysis can be used in signal processing for removing noise. Jean Baptist Joseph Fourier (1807) has introduced frequency analysis leading Fourier analysis. He also explained the fact that functions can be represented as the sum of sine and cosine which led to Fourier Transform.

Alfred Haar in the year 1909 introduced wavelets. Haar's contribution to wavelets is very evident so the entire wavelet family named after him. The Haar wavelets are the simplest of the wavelet families.

II. Haar wavelets [2,3,4]

A Haar wavelet is the simplest type of wavelet. In discrete form, Haar wavelets are related to a mathematical operation called the Haar transform. The Haar transform serves as a prototype for all other wavelet transforms.

III. Haar spaces

Let $\phi(t)$ be the box function defined by,

$$\phi(t)=\begin{Bmatrix}1 & 0\le t<1\\ 0 & otherwise\end{Bmatrix}. \tag{1}$$

The equation,

$$\varphi(t)=\phi(2t)+\phi(2t-1) \tag{2}$$

is called the dilation equation. The Haar function $\varphi(t)$ is typically called a scaling function. The space $V_0 = span\{\phi(t-k)\}_{k\in Z} \cap L^2(R)$ is called the Haar space V_0 generated by the Haar function $\phi(t)$. The set $\{\phi(t-k)\}_{k\in Z}$ forms an orthonormal basis for V_0 . The set $\{\phi(t-k)\}_{k\in Z}$ also forms a Riesz basis for V_0.

[**Riesz basis**: Suppose that $\phi(t)\in L^2(R)$ and $0<A<B$ are constants such that

$$A\sum_{n\in Z}c_n^2 \le \left\| \sum_{n\in Z}c_n\phi_n(t)\right\| \le B\sum_{n\in Z}c_n^2 \tag{3}$$

for any square-summable sequence $\vec{c}$. Then we say that the translates $\{\phi_n(t)\}_{n\in Z}$ form a Riesz basis of $V_0 = span\{\phi_n(t)\}_{n\in Z}$.] The projection of the functions $f(t)\in L^2(R)$ into V_0 is given by,

$$P(f(t))=\sum_{k\in Z}\langle f(t),\, \phi(t-k)\rangle \phi(t-k) \tag{4}$$

The vector space $V_j = span\{\phi(2^j t-k)\}_{k\in Z} \cap L^2(R)$ is called the Haar space V_j .$V_j \subseteq L^2(R)$ is a vector space of piecewise constant functions with possible breakpoints at $2^{-j}Z$. For each $k\in Z$, define the function

$$\phi_{j,k} = 2^{\frac{j}{2}}\phi\left(2^{\frac{j}{2}}t-k\right). \tag{5}$$

The set $\{\phi_{j,k}(t)\}_{k\in Z}$ is an orthonormal basis for V_j . The compact support for the functions $\phi_{j,k}(t)$ is given by,

$$\overline{\text{supp}(\phi_{\text{j,k}})} = \left[\frac{k}{2^j}\ ,\ \frac{k+1}{2^j}\right]. \tag{6}$$

The dyadic interval $I_{j,k}$, $j,k \in Z$, dilation equation and projection function are defined by

$$I_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right) = \left\{t \in R \middle| \frac{k}{2^j} \le t < \frac{k+1}{2^j}\right\}. \tag{7}$$

$$\phi_{j,k}(t) = \frac{\sqrt{2}}{2}\phi_{j+1,2k}(t) + \frac{\sqrt{2}}{2}\phi_{j+1,2k+1}(t) \ . \tag{8}$$

$$P_j(f(t)) = \sum_{k \in Z} \langle \phi_{j,k}(t),\ f(t) \rangle \phi_{j,k}(t), \tag{9}$$

where P_j is called the approximation operator and the space V_j is also known as approximation space. The Haar spaces $\{V_j\}_{j \in Z}$ satisfy

- $\cdots\cdots\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots\cdots\cdots$
- The function $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$.
- $\bigcap_{j \in Z} V_j = \cdots\cdots\cdots \cap V_{-1} \cap V_0 \cap V_1 \cap V_2 \cap \cdots\cdots\cdots = \{0\}$.
- $\overline{\bigcup_{j \in Z} V_j} = \overline{\cdots\cdots\cdots \cup V_{-1} \cup V_0 \cup V_1 \cup V_2 \cup \cdots\cdots\cdots} = L^2(R)$

(10)

IV. Haar wavelet spaces

The wavelet function $\psi(t)$ is given by,

$$\psi(t) = \phi(2t) - \phi(2t-1) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & otherwise \end{cases}. \tag{11}$$

The equation,
$\psi(t) = \phi(2t) - \phi(2t-1)$, is called the dilation equation. The space $W_0 = span\{\psi(t-k)\}_{k \in Z} \cap L^2(R)$ is called the Haar wavelet space W_0 generated by the Haar wavelet function $\psi(t)$. The set $\{\psi(t-k)\}_{k \in Z}$ forms an orthonormal basis for W_0 . The set $\{\psi(t-k)\}_{k \in Z}$ also forms a Riesz basis for W_0 . The Haar wavelet space W_0 satisfies the following properties.

- ♣ If $f(t) \in V_0$ and $g(t) \in W_0$, then $\langle f(t),\ g(t) \rangle = 0$.
- ♣ V_0 and W_0 are perpendicular to each other.
- ♣ $V_1 = V_0 \oplus W_0$.

(12)

The vector space $W_j = span\{\psi(2^j t - k)\}_{k \in Z} \cap L^2(R)$ is called the Haar wavelet space W_j .
For each $k \in Z$, define the function $\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^{j/2}t - k)$. The set $\{\psi_{j,k}(t)\}_{k \in Z}$ is an orthonormal basis for W_j . The compact support for the functions $\psi_{j,k}(t)$ is given by,

$$\overline{\text{supp}(\psi_{\text{j,k}})} = \left[\frac{k}{2^j}\ ,\ \frac{k+1}{2^j}\right]. \tag{13}$$

For $j,k \in Z$, the dilation equation for $\psi_{j,k}(t)$ is given by,

$$\psi_{j,k}(t)=\frac{\sqrt{2}}{2}\phi_{j+1,2k}(t)-\frac{\sqrt{2}}{2}\phi_{j+1,2k+1}(t)\cdot \tag{14}$$

For each $j\in Z$, the detail operator Q_j on functions $f(t)\in L^2(R)$ is defined by,

$$Q_j(f(t))=P_{j+1}(f(t))-P_j(f(t))\cdot \tag{15}$$

The Haar wavelet spaces $\{W_j\}_{j\in Z}$ satisfy the following properties.
If j and l are integers with $l\geq j$, then V_j and W_j are perpendicular to each other.
If j and l are integers with $l\neq j$, then W_j and W_l are perpendicular to each other.
$V_{j+1}=V_j\oplus W_j$.

Let $f_{j+1}(t)\in V_{j+1}$ be defined as $f_{j+1}(t)=\sum_{m\in Z}a_m\phi_{j,m}(t)$ and suppose that $\vec{h}$ is the vector given by $\vec{h}=[h_0,h_1]^T=\left[\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]^T$, then the projection of $f_{j+1}(t)$ into V_j can be written as

$$f_j(t)=\sum_{k\in Z}b_k\phi_{j,k}(t)=\sum_{k\in Z}\left\langle f_{j+1}(t),\,\phi_{j,k}(t)\right\rangle\phi_{j,k}(t) \tag{16}$$

where $b_k=\frac{\sqrt{2}}{2}(a_{2k}+a_{2k+1})=\vec{h}\cdot\vec{a}^{\,k}$ for $k\in Z$ and $\vec{a}^{\,k}=[a_{2k},a_{2k+1}]^T$.

Let $f_{j+1}(t)\in V_{j+1}$ be defined as

$$f_{j+1}(t)=\sum_{m\in Z}a_m\phi_{j+1,m}(t)\cdot \tag{17}$$

Suppose that $f_j(t)$ is the projection of $f_{j+1}(t)$ into V_j and $\vec{g}$ is the vector given by

$$\vec{g}=[g_0,g_1]^T=\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right]^T. \tag{18}$$

If $g_j(t)=f_{j+1}(t)-f_j(t)$ is the residual function in V_{j+1}, then $g_j(t)\in W_j$ and is thus given by

$$g_j(t)=\sum_{k\in Z}c_k\psi_{j,k}(t) \tag{19}$$

where $c_k=\frac{\sqrt{2}}{2}(a_{2k}-a_{2k+1})=\vec{g}\cdot\vec{a}^{\,k}$ for $k\in Z$ and $\vec{a}^{\,k}=[a_{2k},a_{2k+1}]^T$.

Thus, a piecewise constant function $f_j(t),\ j>0$ with possible breakpoints at points in 2^jZ can be decomposed into a piecewise constant approximation function $f_{j-1}(t)$ with possible breakpoints on the coarser grid $2^{j-1}Z$ and a piecewise constant detail function $g_{j-1}(t)$ with possible breakpoints in 2^jZ. This process can be iterated and finally $f_j(t)$ can be written as the sum of an approximation function $f_0(t)$ with possible breakpoints at the integers and detail functions $g_0(t),\,g_1(t),\cdots\cdots,\,g_{j-1}(t)$ with possible breakpoints at $\frac{1}{2}Z,\,\frac{1}{4}Z,\cdots\cdots,\,2^jZ$ respectively. The next step is to model the decomposition in terms of linear transformations (matrices). The technique to process discrete data via a discrete version of the decomposition process is called the discrete Haar wavelet transformation.

V. Discrete Haar wavelet transformation

Suppose that N is an even positive integer. The discrete Haar wavelet transformation is defined as,

$$W_N = \left[\frac{H_{N/2}}{G_{N/2}}\right] = \left[\begin{array}{ccccccc} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \hline \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{array}\right] \quad (20)$$

The $\frac{N}{2} \times N$ block $H_{N/2}$ is called the averages block and the $\frac{N}{2} \times N$ block $G_{N/2}$ is called the details block. We have, $H_{N/2}\vec{a} = \vec{b}$ and $G_{N/2}\vec{a} = \vec{c}$. The inverse discrete Haar wavelet transformation is given by,

$$\vec{a} = W_N^T \left[\frac{\vec{b}}{\vec{c}}\right]. \quad (21)$$

The two-dimensional discrete Haar wavelet transformation B of a matrix A is defines as,

$$B = W_M A W_N^T \cdot$$

The inverse of two-dimensional discrete Haar wavelet transformation is given by,

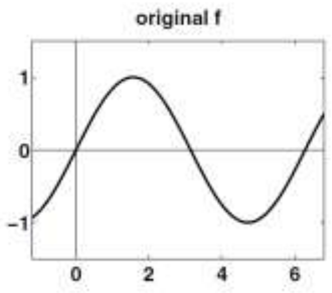
$$A = W_M^T B W_N \cdot$$

VI. Advantages of Haar Wavelet Transform

a) It is very simple.
b) It is fast.
c) It can be calculated without any need of temporary array so it needs less memory.
d) It can be reversed exactly without the edge effects.

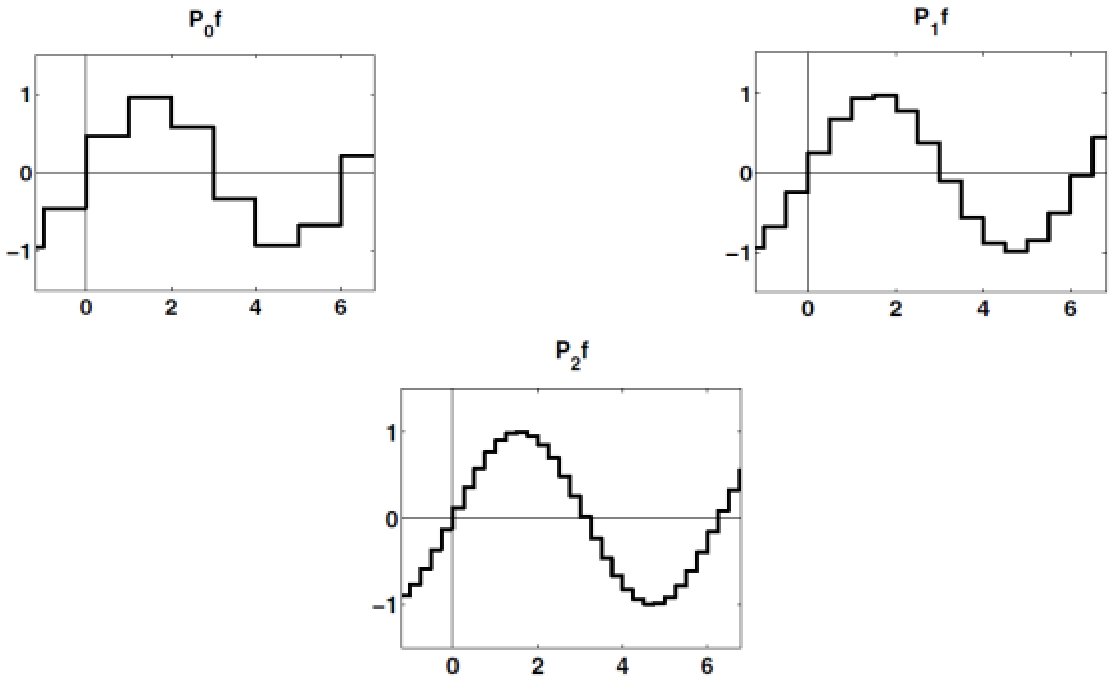
VII. GRAPHS [5]

Original function $f = \sin x$



Haar Wavelet

Haar Wavelet for $f = \sin x$



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Refernces

[1.] A brief history of Wavelets – Brendon Keift - Internet
[2.] A Primer on WAVELETS and their Scientific Applications – James S. Walker –Chapman & Hall/CRC.
[3.] WAVELET THEORY: An Elementary Approach with Applications – David K. Ruch & Patrick J. Van Fleet – Wiley.
[4.] An Introduction to Wavelet Analysis – David F. Walnut – Birkhäuser.
[5.] Wavelets and Multiwavelets – Fritz Keinert – Chapman & Hall/CRC.
[6.] Wavelets: Tools for Science and Technology – Stephane Jaffard, Yves Meyer, Robert D. Ryan – Siam.
[7.] THE WORLD According to WAVELETS: The Story of a Mathematical Technique in the Making – Barbara Burke Hubbard – Universities Press.