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Frequency Analysis of aFree Vibrating SSCC Thin Rectangular Orthotropic Plate Using Improved Rayleigh's Method.

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Abstract: This paper presents the solution of the analysis of free vibration of a rectangular thin orthotropic plate using an improved Rayleigh's method. The plate is bounded by two adjacent simply supported edges (i.e. SS) and another two adjacent clamped edges (i.e. CC). The total potential energy functional of a free vibrating plate, was derived from first principle, using the theory of elasticity. A truncated Taylor's Mclaurin series of fourth terms was used to develop a general deflection function that satisfies the boundary conditions of the given plate. The deflection function was then, substituted into the potential energy functional and the resulting equation subsequently minimized. Thereafter, the equation for natural frequency, λ, of the SSCC plate was determined, and used to obtain natural frequencies for aspect ratios ranging from 0.1 to 2.0, in steps of 0.1. These average percentage differences, indicate that the formulated deflection function for the clamped plate, is a very good approximation to the exact deflection function of the free vibration of a clamped rectangular thin orthotropic plate.
Keywords: Clamped Rectangular Plate, Free Vibration Analysis, Improved Rayleigh's Method, Natural Frequency, Orthotropic Vibrating Plate, and Shape Function.

I. INTRODUCTION

[1] Obtained the exact solution of large deflections analysis of clamped circular plates. [2]Used Ritz method in the analysis of plated with opposite sides simply supported and other possible combinations of clamped, simply supported and free edge conditions and presented their analytical results. [3], conducted the free vibration analysis of isotropic and anisotropic rectangular thin plates subjected to general boundary conditions using a modified Ritz method. [4] Applied the method of superposition to the vibration analysis of rectangular plates with a combination of clamped and simply supported boundary conditions. [5], using novel separation of variables, obtained the exact solutions for free vibrations of rectangular thin orthotropic plates with all combinations of simply supported and clamped boundary conditions. One of the plate cases he considered is the SSCC plate. [6], using Taylor's series function in Rayleigh-Ritz method, obtained a new approximate solution of SSCC plate.

II. THEORITICAL FORMULATION

2.1 Differential Equation of a Thin Rectangular Orthotropic Plate in Vibration.

[5] Derived the following governing differential equation of a thin orthotropic plate experiencing free vibration:

$$D_1\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2D_3\frac{\partial^4 w(x,y,t)}{\partial x^2 dy^2} + D_2\frac{\partial^4 w(x,y,t)}{\partial y^4}\rho h\frac{\partial^2 w}{\partial t^2}(x,y,t) = 0 \qquad (1)$$

WhereD_1, D_2 and D_3 are flexural rigidities of the plate$W(x,y,t)$ is the deflection function of the plate
x and y are cartesian co-ordinate of the plate.
t = thickness of the plate.
ρ = density of the material
h = plate thickness

Using Taylor –Maclaurin series, [7] expressed the shape function, w as follows:

$$W = W(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{F^{(m)}(x_0)F^{(n)}(y_0)}{m!\,n!}(x-x_0)^m.(y-y_0)^n \quad (2)$$

Where$F^{(m)}(x_0)$ is the m^{th} partial derivative of the function, w, with respect to x.$F^{(n)}(y_0)$is the n^{th} partial derivative of the function, w, with respect to y, m! and n! are the factorials of m and n respectivelyx_0 and y_0 are the points of origin.
By truncating the infinite series at m= n =4, the Equation (2) reduces to Equation (3)

$W = \sum_{m=0}^{1}\sum_{n=0}^{4} I_m J_n \, x^m . y^n$ (3)

Expressing Equation (3) in terms of non-dimensional co-ordinates, R and Q, yields Equation (4)

$w = \sum_{m=0}^{4}\sum_{n=0}^{4} a_m b_n \, R^m Q^n$ (4)

where

$a_m = I_m . a^m$ (5)

and

$b_n = J_n . b^n$ (6)

But R = x/a and Q = y/b (7)

The function given by Equation (4) can be further expanded in the following form:

$w(R,Q) = (a_0 + a_1R + a_2R^2 + a_3R^3 + a_4R^4)(b_0 + b_1Q + b_2Q^2 + b_3Q^3 + b_4Q^4)$ (8)

Wherea_i and b_i (i= 0,1,2,3,4) are constants.

2.2 Boundary Conditions of SSCC Orthotropic Plate.

The thin rectangular orthotropic plate considered in this work, has two adjacent simply supported edges and another two adjacent clamped edges as shown in Fig 1.

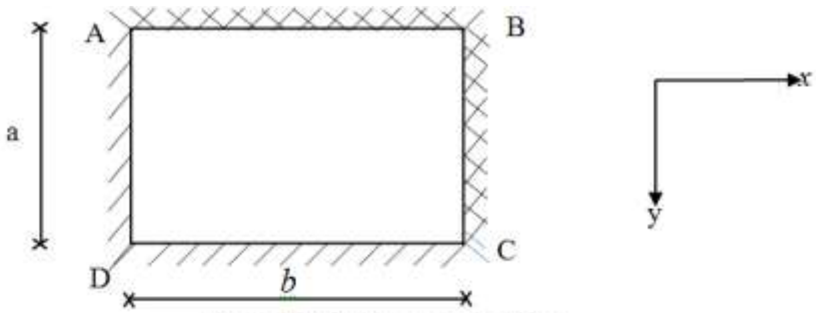

Fig 1: SSCC Rectangular Plate

The edges AD and DC are simply supported, while edges AB and BC are clamped.

For simply supported edges, the deflections and bending moment vanishes. The boundary conditions for the SSCC plate, are as follows:

$W(R=0) = 0$; $W'^{R}(R=0) = 0$

$W(R=1) = 0$; $W'^{R}(R=1) = 0$

$W(Q=0) = 0$; $W'^{Q}(Q=0) = 0$

$W(Q=1) = 0$; $W''^{Q}(Q=1) = 0$

Substituting successively, the boundary conditions, $\mathbf{W(R=0)=0}$,$\mathbf{W^{R}(R=0)=0}$, $\mathbf{W(Q=0)=0}$, and $\mathbf{W^{Q}(Q=0)=0}$ into the Equation (8) yields:

$a_0 = 0$ (9)

$a_1 = 0$ (10)

$b_0 = 0$ (11)

$b_1 = 0$ (12)

Similarly, substituting successively, the boundary conditions, $\mathbf{W(Q=1)=0}$ and $\mathbf{W^{Q}(Q=1)=0}$ into Equation (8), gives respectively:

$a_2 + a_3 + a_4 = 0$ (13)

$2a_2 + 6a_3 + 12a_4 = 0$ (14)

Solving Equations (13) and (14), yields the following:

$a_1 = 1.5a_4$anda_3 =2.5 b_4 (15)

Substituting these values of a_1, and a_2, into Equation (8), gives the following displacement function, W.

W =$a_4b_4(1.5R_2 - 2.5R_3 + R_4)(1.5Q_2 - 2.5Q_3 + Q_4)$ (16)

W = AH (17)

where:
A is the amplitude of the deflected shape=a_4b_4 (18)
H is the deflected shape = $(1.5R_2 - 2.5R_3 + R_4)(1.5Q_2 - 2.5Q_3 + Q_4)$ (19)

2.3 Application of Variational Principle.
The differentiation of the partial derivatives of the deflected shape, H, with respect to the dimensionless parameters, R and Q, are as follows:

$W'^{R} = A\frac{\partial H}{\partial R}$ (20)

$W''^{R} = A\frac{\partial^2 H}{\partial R^2}$ (21)

$W'^{Q} = A\frac{\partial H}{\partial Q}$ (22)

$W''^{Q} = H\frac{\partial^2 H}{\partial Q^2}$ (23)

$W''^{RQ} = A\frac{\partial^2 H}{\partial R\,\partial Q}$ (24)

Squaring and integrating partially the Equations (17), (21),(23) and (24) with respect to R and Q in close domain, yields equations (25),(26),(27) and (28) respectively.

$\int_0^1\int_0^1 (AH)^2\,\partial R\,\partial Q = 0.0000056846813\,A^2$ (25)

$\int_0^1\int_0^1 (A\frac{\partial^2 H}{\partial R^2})^2\,\partial R\,\partial Q = 0.013572A^2$ (26)

$\int_0^1\int_0^1 (A\frac{\partial^2 H}{\partial Q^2})^2\,\partial R\,\partial Q = 0.013572A^2$ (27)

$\int_0^1\int_0^1 (A\frac{\partial^2 H}{\partial R\,\partial Q})^2\,\partial R\,\partial Q = 0.0073469A^2$ (28)

III.FORMULATION OF NATURAL FREQUENCY EQUATION FOR A VIBRATING THIN RECTANGULAR ORTHOTROPIC PLATE.

The total potential energy functional, Π, is given by Equation (29)
U = Π – KE (29)
where U = strain energy
KE = Kinetic Energy. But,

$U = \frac{D_x}{2b^2}\int_0^1\int_0^1[\frac{\varphi_1}{p^3}(H\frac{\partial^2 H}{\partial R^2})^2 + 2\frac{\varphi_2}{p}(H\frac{\partial^2 H}{\partial R\,\partial Q})^2 + p\varphi_3(H\frac{\partial^2 H}{\partial Q^2})^2]\partial R\,\partial Q$ (30)

And, the kinetic energy, KE, is given as:

$K.E = \frac{pb^2\lambda^2\rho t}{2}\int_0^1\int_0^1 (AH)^2\,\partial R\,\partial Q$ (31)

where P = aspect ratio
λ = natural frequency

Substituting the value of strain energy U and kinetic energy, KE into Equation (29) gives Equation (32):

$\Pi = \frac{D_x}{2b^2}\int_0^1\int_0^1[\frac{\varphi_1}{p^3}(H\frac{\partial^2 H}{\partial R^2})^2 + 2\frac{\varphi_2}{p}(H\frac{\partial^2 H}{\partial R\,\partial Q})^2 + p\varphi_3(H\frac{\partial^2 H}{\partial Q^2})^2]\partial R\,\partial Q - \frac{pb^2\lambda^2\rho t}{2}\int_0^1\int_0^1 (AH)^2\,\partial R\,\partial Q$ (32)

where

$\varphi_1 = \frac{D_x}{D_x} = 1$ (33)

$\varphi_2 = \frac{B}{D_x}$ (34)

and

$\varphi_3 = \frac{D_y}{D_x}$ (35)

Minimizing the Equation (32) yields:

$$\frac{\partial \Pi}{\partial A} = \frac{D_x A}{b^2}\int_0^1\int_0^1[\frac{\varphi_1}{p^3}(\frac{\partial^2 H}{\partial R^2})^2 + 2\frac{\varphi_2}{p}(\frac{\partial^2 H}{\partial R\,\partial Q})^2 + p\varphi_3(\frac{\partial^2 H}{\partial Q^2})^2]\,\partial R\,\partial Q - pAb^2\lambda^2\rho t\int_0^1\int_0^1 (H)^2\,\partial R\,\partial Q \quad (36)$$

Rearranging the Equation (36), gives natural frequency squared,λ^2, for different aspect ratios, p
(a) For aspect ratio, $= {}^{a}/_{b}$, the square of the natural frequency, λ^2, are given as follows:
(i) λ^2 in terms of p and b, yields Equation (37)

$$\lambda^2 = \frac{\frac{D_x}{b^4\rho t}\int_0^1\int_0^1[\frac{\varphi_1}{p^4}\left(\frac{\partial^2 H}{\partial R^2}\right)^2 + \frac{2\varphi_2}{p^2}\left(\frac{\partial^2 H}{\partial R\,\partial Q}\right)^2 + \varphi_3\left[\left(\frac{\partial^2 H}{\partial Q^2}\right)^2\right]\partial R\,\partial Q}{\int_0^1\int_0^1 H^2\,\partial R\,\partial Q} \quad (37)$$

(ii) λ^2 in terms of a and b, gives Equation (38)

$$\lambda^2 = \frac{\frac{D_x}{a^4\rho t}\int_0^1\int_0^1[\varphi_1\left(\frac{\partial^2 H}{\partial R^2}\right)^2 + 2\varphi_2 a^2\left(\frac{\partial^2 H}{\partial R\,\partial Q}\right)^2 + \varphi_3 a^4\left[\left(\frac{\partial^2 H}{\partial Q^2}\right)^2\right]\partial R\,\partial Q}{b^2} \quad (38)$$

(iii) λ^2 in terms of a and b, yields Equation (39)

$$\lambda^2 = \frac{D_x}{a^4\rho t}\int_0^1\int_0^1[\varphi_1\left(\frac{\partial^2 H}{\partial R^2}\right)^2 + 2\varphi_2 a^2\left(\frac{\partial^2 H}{\partial R\,\partial Q}\right)^2 + \varphi_3 a^4\left[\left(\frac{\partial^2 H}{\partial Q^2}\right)^2\right]\partial R\,\partial Q \quad (39)$$

(b) For aspect ratio, $p = b/a$, the square of the natural frequency, λ^2, is given by Equation (40).

(i) λ^2 $intermsofpandbgivesequation$ (40)

$$\lambda^2 = \frac{\frac{D_x}{a^4\rho t}\int_0^1\int_0^1[\varphi_1\left(\frac{\partial^2 H}{\partial R^2}\right)^2 + \frac{2\varphi_2}{p^2}\left(\frac{\partial^2 H}{\partial R\,\partial Q}\right)^2 + \frac{\varphi_3}{p^4}\left[\left(\frac{\partial^2 H}{\partial Q^2}\right)^2\right]\partial R\,\partial Q}{\int_0^1\int_0^1 H^2\,\partial R\,\partial Q} \quad (40)$$

Substituting the relevant Equations (25) – (28) into the Equation (37) and simplifying the resulting Equation yields Equation (41)

$$\lambda^2 = \frac{D_x}{b^4\rho t}\left[\frac{\varphi_1}{p^4} * 238.746 + \frac{\varphi_2}{p^3} * 258.481 + \varphi_3 * 238.746\right]; \quad \text{for } p = \frac{a}{b} \quad (41)$$

In terms of a and p, the Equation (41) becomes Equation (42)

$$\lambda^2 = \frac{D_x}{a^4\rho t}\left[\varphi_1 * 238.746 + \varphi_2 * 258.481p^2 + \varphi_3 * 238.746p^4\right]; \quad \text{for } p = \frac{a}{b} \quad (42)$$

In terms of 'a' and 'b', Equation (41) represented as Equation (43)

$$\lambda^2 = \frac{D_x}{a^4\rho t}\left[\varphi_1 * 238.746 + \frac{\varphi_2 * a^2}{b^2} * 258.481 + \frac{\varphi_3 a^4}{b^4} * 238.746\right]; \text{ for } p = \frac{a}{b} \quad (43)$$

Then, for the reciprocal of the aspect ratio (i.e) $p = b/a$, the square of the fundamental frequency is given by equation (44)

$$\lambda^2 = \frac{Dx}{a^4 et}\left[\varphi_1 * 238.746 + \frac{\varphi_4}{p^2} * 258.481 + \frac{\varphi_3}{p^4} * 238.746\right]; \text{ for } p = b/a \quad (44)$$

where $\varphi_1 = \frac{D_x}{D_x}, \varphi_2 = \frac{B}{D_x}, \varphi_3 = \frac{D_y}{D_x}$

The fundamental frequencies, λ_i, are the roots of Equations (41)-(44). These fundamental frequencies, λ, of an SSCC plate, can be obtained for various value ofaspect rations, $p = b/a$ and combinations of flexural rigidities, φ_1, φ_2 and φ_3.

However, the exact solution can be obtained from the following expression given by Xing and Liu (2009).

$$D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2\partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} - \beta^4 w = 0$$

IV. RESULTS AND DISCUSSION

The Equations (42) and (44), were used to determine the fundamental frequencies, λ, for various aspect ratios, $p = b/a$ and different combinations of flexural rigidities, φ_1, φ_2 and φ_3 of an SSCC thin rectangular orthotropic plate undergoing free vibration. The results obtained are presented in Tables 1-3. Also the solution by [5] and[8], for aspect ratios, ($p = b/a$) of 0.5, 1.0 and 2.0, are given in the same Tables 1-3.

A comparison of the new solution in this work with those of [5], shows that the average percentage differences are 0.7791%, 0.5702% and 0.6440% for aspect ratios ($p = b/a$) of 0.5, 1.0 and 2.0. A similar comparison with the solutions by [8] yields relatively, smaller average percentage differences for the same aspect ratios of 0.5, 1.0 and 2.0 than the corresponding average percentage differences between the results of this work and those of exact solution. Thus, there is a very close agreement between the new fundamental frequency results obtained and those of [5], and [8].

Besides, the graph of fundamental frequencies, λ , and aspect ratios, p= $^b/_a$, (given in Tables 1-3), are plotted for various combinations of flexural rigidities φ1 φ 2 and φ3 (see Fig 2). The diagram shows that there is high convergence of the three curves as the aspect ratio increased.

Table 1: Fundamental frequencies, λ ,of a free vibrating SSCC plate for various aspect ratios, p= $^b/_a$, and flexural rigidities, φ 1= φ 2= φ 3= 1

Aspect Ratios, p= $^b/_a$,	New solution		Exact solution, λ_2	Kantorovich's Solution, λ_3
	λ_1^2	λ_1		
0.1	2413451	1553.528		
0.2	155911	394.856		
0.3	32584.37	180.511		
0.4	11179.88	105.735		
0.5	5092.441	71.361	70.877	71.081
0.6	2798.84	52.904		
0.7	1760.569	41.959		
0.8	1225.465	35.007		
0.9	921.7203	30.360		
1	735.9535	27.128	26.867	27.059
1.1	615.417	24.808		
1.2	533.3682	23.095		
1.3	475.2718	21.801		
1.4	432.7593	20.803		
1.5	400.7745	20.019		
1.6	376.1337	19.394		
1.7	356.7601	18.888		
1.8	341.2564	18.473		
1.9	328.6568	18.129		
2	318.2776	17.840	17.719	17.770

where λ= fundamental frequency , λ2= fundamental frequency squared.

Table 2: Fundamental frequencies, λ, of a free vibrating SSCC plate for various aspect ratios, p= $^b/_a$ and flexural rigidities, φ =φ3=1 and φ2=0.5

Aspect ratios, P=b/a	New solution		Exact solution, λ_2	Kantorovich' Solution, λ_3
	λ_1^2	λ_1		
0.1	2400527	1549.363		
0.2	152680	390.743		
0.3	31148.37	176.489		
0.4	10372.13	101.844		
0.5	4575.48	67.642	67.331	67.497
0.6	2439.839	49.395		
0.7	1496.813	38.689		
0.8	1023.527	31.993		
0.9	762.1643	27.601		
1	606.7131	24.632	24.449	24.610
1.1	508.6068	22.552		
1.2	443.618	21.062		
1.3	398.7982	19.970		
1.4	366.8204	19.153		
1.5	343.3344	18.529		
1.6	325.6492	18.046		
1.7	312.0402	17.665		
1.8	301.3674	17.360		
1.9	292.8561	17.113		
2	285.9675	16.911	16.833	16.874

where λ= fundamental frequency , λ2= fundamental frequency squared.

Table 3: Fundamental frequencies, λ, of a free vibrating SSCC plate for various aspect ratios, p= $^b/_a$ and flexural rigidities, φ =1, φ3=0.5 and φ2=0.5.

Aspect Ratios, p= $^b/_a$,	New solution		Exact solution, λ_2	Kantorovich's Solution, λ_3
	λ_1^2	λ_1		
0.1	1206845	1098.565		
0.2	78074.87	279.419		
0.3	16411.55	128.108		
0.4	5709.309	75.560		
0.5	2665.589	51.629	51.302	51.507
0.6	1518.788	38.972		
0.7	999.6525	31.617		

0.8	732.1007	27.057		
0.9	580.2284	24.088		
1	487.3449	22.076	21.898	22.042
1.1	427.0767	20.666		
1.2	386.0523	19.648		
1.3	357.0041	18.895		
1.4	335.7479	18.323		
1.5	319.7554	17.882		
1.6	307.435	17.534		
1.7	297.7482	17.255		
1.8	289.9964	17.029		
1.9	283.6966	16.843		
2	278.507	16.689	16.609	16.638

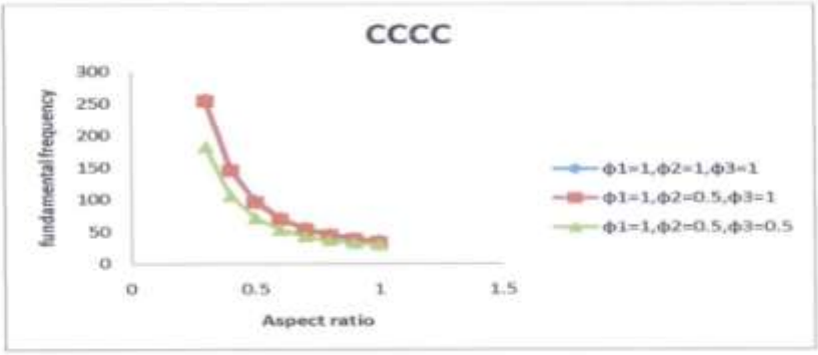

Fig 2: Graph of fundamental frequency, λ against aspect ratio, p

V. CONCLUSION

The closeness of the fundamental frequencies, λ, from the three methods, underscores the similarity in the deflection functions chosen in the three methods. The new equations formulated in this work, can be used to compute very close approximation of the fundamental frequencies of an SSCC thin rectangular orthotropic plate undergoing vibration. And the newly formulated equations, give upper bound values of the fundamental frequencies.

The convergence of the three curves given in Fig 2, indicates that the fundamental frequency at a certain value of the aspect ratio (i.e. at about $p = {}^{b}/_{a} = 1$), becomes approximately constant, irrespective of the combination of the flexural rigidities, φ 1 , φ 2 and φ 3.

And, the use of Taylor's series in Rayleigh- Ritz method, overcomes the limitations encountered in the derivation of fundamental frequency using conventional methods.

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